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MINIMAL TIME SEQUENTIAL BATCH REACTORS WITH BOUNDED AND IMPULSE CONTROLS FOR ONE OR MORE SPECIES*

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Abstract. We consider the optimal control problem of feeding in minimal time a tank where several species compete for a single resource, with the objective being to reach a given level of the resource. We allow controls to be bounded measurable functions of time plus possible impulses. For the one-species case, we show that the immediate one-impulse strategy (filling the whole reactor with one single impulse at the initial time) is optimal when the growth function is monotonic. For nonmonotonic growth functions with one maximum, we show that a particular singular arc strategy (precisely defined in section 3) is optimal. These results extend and improve former ones obtained for the class of measurable controls only. For the two-species case with monotonic growth functions, we give conditions under which the immediate one-impulse strategy is optimal. We also give optimality conditions for the singular arc strategy (at a level that depends on the initial condition) to be optimal. The possibility for the immediate one-impulse strategy to be nonoptimal while both growth functions are monotonic is a surprising result and is illustrated with the help of numerical simulations.

Key words. minimal time problem, chemostat, Hamilton–Jacobi–Bellman equation, Pontryagin's minimum principle, impulse control

AMS subject classifications. 49J15, 49N25

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1. Introduction.

Sequential batch reactors (SBR) are often used in biotechnological industries, notably in waste-water treatment. Typically, a tank is filled with activated sludge or biological microorganisms capable of degrading some undesirable substrate. The method then consists of a sequence of cycles composed of three phases:

- Phase 1: Filling the reactor with water to be treated,
- Phase 2: Waiting for the concentration of the undesirable substrate to decrease until a given (low) concentration,
- Phase 3: Emptying the *clean* water from the reactor, leaving the sludge inside.

The time necessary to achieve such cycles can be substantially long and can have an economic impact on the overall process. Manipulating the input flow during the filling phase clearly has an influence on the total duration of the cycle (more precisely, the duration of Phases 1 and 2, the duration of Phase 3 being fixed). But the nonlinear kinetics of the biological reactions do not always make easy the determination of the input flow strategy that minimizes the total time obvious.

Very similar problems (optimizing the production of biomass at a fixed terminal time) have already been tackled with the help of optimal control theory [1, 11, 10],

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which has led to computational methods [30, 12]. For models with one biological species, a solution of the minimal time problem has been proposed by Moreno in [20] for monotonic as well as nonmonotonic kinetics. It has been proved that, for monotonic growth functions, such as the Monod law (see [29]), the optimal solution consists of a *most rapid approach* strategy, namely, filling the tank up to its maximum capacity as fast as possible and then waiting. For nonmonotonic growths with one maximum, such as the Haldane law (see [29]), a *singular arc* strategy which consists of maintaining the resource level that maximizes the growth function for most of the time, have been proved to be optimal. The optimality proofs are based on a technique due to Miele [13] using Green's theorem. More precisely, proofs rely on a reformulation of the problem in a planar one.

In the present work, we consider minimal time problems where more than one species can compete for the same substrate. For these cases, the problem cannot be reformulated into a planar one, and the technique mentioned above does not apply. Nevertheless, we are interested in characterizing biological systems for which the *most rapid approach* strategy is again optimal. We are also interested in identifying conditions for which a *singular arc* strategy could be optimal.

We shall allow adding unbounded or impulsive controls to the usual measurable bounded controls. The practical motivation for such a consideration comes from the fact that a bounded measurable control can be incorporated into a device that tunes the speed of a pump over a certain range, while an unbounded control can be assimilated to an instantaneous dilution of a positive volume, as in [9]. A similar optimal control problem for fed-batch processes has been studied in [32] but for a fixed terminal time and a final cost. A characterization of minimal time functions with impulse controls and state constraints has been proposed in [7, 26]. In [7], some restrictive conditions are considered on the jumps that do not apply to the present problem. In [26], the minimal time function is characterized but as a function of a bound on the total variation allowed on the unbounded control. For related results concerning the regularity of the value function for minimal time problems, see [24] and the references therein.

For our problem with a scalar control, we use a smooth time parameterization in the spirit of [33] and [34], which differs from more general approaches that use discontinuous time transformation (see, for instance, [3, 8, 14, 15, 16, 18, 19] or [35]). The possibility of immediately reaching the target with a single jump has also led us to extend the definition of the singular arc strategy to the framework of impulse controls.

Even though the main contribution of this paper is the analysis of the two species case, it is worth noting that the former results of Moreno [20] for the one-species case without impulse controls did not consider the parametric configuration $s^\star < s_{out}$ (the notation will be defined in section 3). This case leads to more complicated optimal trajectories, as we shall show. Furthermore, we provide an explicit expression for the value function for any parametric configuration.

The paper is organized as follows. In the next section, we state the minimal time problem with impulse control and give an equivalent formulation with measurable controls. In section 3, we define the *one-impulse* and *singular arc* strategies. Section 4 characterizes the cost of the *one-impulse* strategy, which plays an important role in the following sections. Section 5 gives the Hamilton–Jacobi formulation of the problem and states optimality results for the strategies presented in section 3. The use of the minimum principle is presented in section 6. Finally, applications to the one- and two-species cases are given in sections 7 and 8, respectively.

2. Formulation of the problem. The dynamics of an SBR with several species can be described by the following set of ordinary differential equations (see [29]):

$$\begin{cases} \dot{x}_i = \mu_i(s)x_i - \dfrac{F}{v}x_i, \quad x_i(t_0) = y_i \quad (i = 1 \cdots n), \\ \dot{s} = -\displaystyle\sum_{j=1}^{n} \mu_j(s)x_j + \dfrac{F}{v}(s_{in} - s), \quad s(t_0) = z, \\ \dot{v} = F, \quad v(t_0) = w, \end{cases} \tag{2.1}$$

where x_i, s, and v stand, respectively, for the concentration of the ith species, the concentration of the substrate, and the current volume of water present in the tank. The parameter $s_{in} > 0$ is a constant which represents the substrate concentration in the input flow. The growth functions $\mu_i(\cdot)$ are nonnegative smooth functions such that $\mu_i(0) = 0$, and the input flow F is a nonnegative control variable.

Given a (desirable) substrate concentration $s_{out} \in]0, s_{in}[$ and a volume (of the reactor) $v_{\max} > 0$, consider the domain $\mathcal{D} = (\mathbb{R}_+^n \setminus \{0\}) \times]0, s_{in}] \times]0, v_{\max}[$ and the target $\mathcal{T} = \mathbb{R}_+^n \times]0, s_{out}] \times \{v_{\max}\}$. From any initial condition $\xi = (y, z, w)$ in $\mathcal{D}$ at time t_0, the objective is to reach $\mathcal{T}$ in minimal time. Let us write $V(\cdot)$ the value function of the problem

$$V(\xi) = \inf_{F(\cdot)} \left\{ t - t_0 \,|\, s^{t_0,\xi,F}(t) \le s_{out} \ , \ v^{t_0,\xi,F}(t) = v_{\max} \right\}, \tag{2.2}$$

where $s^{t_0,\xi,F}(\cdot)$, $v^{t_0,\xi,F}(\cdot)$ denote solutions of (2.1), with initial condition $\xi \in \mathcal{D}$ at time t_0 and control $F(\cdot)$.

We allow here $F(\cdot)$ to be a nonnegative measurable function plus possible positive impulses. The question of the proper treatment of optimal control problems with unbounded or *impulse* controls has already been studied in the literature (see [5, 6, 8, 14, 15, 17, 18, 19, 21, 22, 23, 26, 27, 35]). Instead of an ordinary control $F(\cdot)$, we consider a measure $dF(\cdot)$ that we decompose into a sum of a measure absolutely continuous with respect to the Lebesgue measure $u(t)dt$ and a singular or *impulsive* part $d\sigma$ (see [33, 34]):

$$dF(t) = u(t)dt + d\sigma. \tag{2.3}$$

Here, $u(\cdot)$ is a measurable nonnegative control that we impose to be bounded from above by $u_{\max}$, because it corresponds to the use of a *pump* device. At time t, the nonnegative *impulse* $d\sigma$ corresponds to an (instantaneous) addition of volume from $v^-(t)$ to $v^+(t)$. When $d\sigma$ is nonnull, it implies that the concentrations x_i and s jump as follows:

$$\begin{cases} x_i^+(t) = x_i^-(t)\dfrac{v^-(t)}{v^+(t)}, \\ s^+(t) = s^-(t)\dfrac{v^-(t)}{v^+(t)} + s_{in}\left(1 - \dfrac{v^-(t)}{v^+(t)}\right). \end{cases}$$

Notice that such a jump is equivalent to integrate the dynamics

$$\frac{dx_i}{d\tau} = -\frac{u}{v}x_i, \quad \frac{ds}{d\tau} = \frac{u}{v}(s_{in} - s), \quad \frac{dv}{d\tau} = u, \tag{2.4}$$

from τ^- to τ^+, with any regular nonnegative control $u(\cdot)$ bounded from above by $u_{\max}$, provided that the integral constraint is fulfilled:

$$\int_{\tau^-}^{\tau^+} u(\tau)d\tau = v^+(t) - v^-(t). \tag{2.5}$$

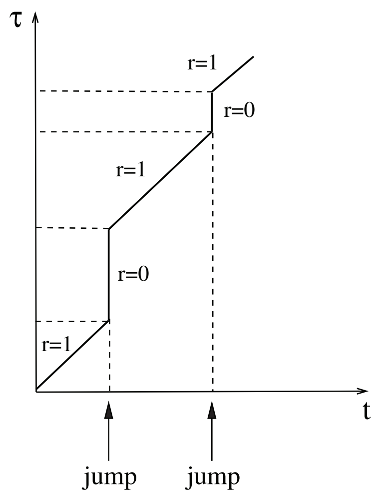

FIG. 2.1. *Time parameterization.*

Consider then a time parameterization $\tau \geq t_0$ such that $dt = r(\tau)d\tau$ (see Figure 2.1), where

$$r(\tau) = \begin{cases} 1 & \text{when } dF \text{ is absolutely continuous (a.c.) at } t(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

Then, dynamics (2.1) with dF regular, and dynamics (2.4) with nonnull $d\sigma$ can be gathered into the system

$$\begin{cases} \dfrac{dx_i}{d\tau} = r\mu_i(s)x_i - \dfrac{u}{v}x_i \quad (i = 1 \cdots n), \\ \dfrac{ds}{d\tau} = -r\displaystyle\sum_{j=1}^{n} \mu_j(s)x_j + \dfrac{u}{v}(s_{in} - s), \\ \dfrac{dv}{d\tau} = u, \end{cases} \tag{2.6}$$

where the controls $u(\cdot)$ and $r(\cdot)$ are sought among measurable functions w.r.t. τ, taking values in $[0, u_{\max}]$ and $\{0, 1\}$, respectively. Notice that, in this formulation, $u(\cdot)$ plays both the role of an ordinary control when $r = 1$ and the control of the amplitude of the jump (2.5) when $r = 0$, with the same single constraint $u \in [0, u_{\max}]$.

Remark 1. We could have considered two distinct controls, as in [18], for instance, if we write the system (2.1) as

$$\dot{X}(t) = f(X(t)) + F(t)g(X(t)),$$

where $X = (x_i, s, v)$, $i = 1, \ldots, n$ and functions f and g are suitably chosen to be compatible with the dynamics (2.1), we could consider

$$\dot{X}(t) = f(X(t)) + (u_1(t) + u_2(t))g(X(t)),$$

with u_1 a measurable nonnegative bounded (by $u_{\max}$) control and u_2 is an unbounded nonnegative control.

In this framework, the time reparametrization $dt = rd\tau$, with $r \in [0,1]$, leads to the dynamics

$$\frac{dX}{d\tau} = r(\tau)[f(X(\tau)) + u_1(\tau)g(X(\tau))] + (1 - r(\tau))w_2(\tau)g(X(\tau)), \tag{2.7}$$

with u_1 and w_2 nonnegative bounded controls. One can also require w_2 to be bounded by the same bounds $u_{\max}$ as u_1, where $u_1(\tau)r(\tau)d\tau = u_1(t)dt$ and $(1-r(\tau))w_2(\tau)d\tau = du_2(t)$. The dynamics (2.7) is equivalent to

$$\frac{dX}{d\tau} = r(\tau)f(X(\tau)) + u(\tau)g(X(\tau)),$$

where $u = ru_1 + (1 - r)w_2$ belongs to $[0, u_{\max}]$.

Remark 2. Since one can always take $r = 0$ and $u = 0$ on an arbitrarily large τ-interval without modifying the total time $\int_{t_0}^{\tau} r(\theta)d\theta$, the minimal time problem has no unique solution. Hence, without loss of generality, we will be only interested in controls that never take null values simultaneously, that is, satisfying $r(\tau) \neq 0$ or $u(\tau) \neq 0$ for all time τ.

Let us define the set of admissible controls by

$$\mathcal{C} = \{(u,r) : [0,+\infty) \mapsto [0,u_{\max}] \times \{0,1\} \setminus \{(0,0)\} \text{ Lebesgue measurable}\}, \tag{2.8}$$

and let us write now $V(\cdot)$ the value function of the reformulated problem (2.6)

$$V(\xi) = \inf_{(u,r)(\cdot)\in\mathcal{C}} \left\{ \int_{t_0}^{\tau} r(\theta)d\theta \,|\, s^{t_0,\xi,u,r}(\tau) \leq s_{out} \ , \ v^{t_0,\xi,u,r}(\tau) = v_{\max} \right\}, \tag{2.9}$$

where $s^{t_0,\xi,u,r}(\cdot)$, $v^{t_0,\xi,u,r}(\cdot)$ denote solutions of (2.6), with initial condition $\xi \in \mathcal{D}$ at time t_0 and controls $u(\cdot)$ and $r(\cdot)$.

Remark 3. Any trajectory of the dynamics (2.6) with initial condition $\xi = (y,z,w) \in \mathcal{D}$ lies in the region defined by

$$\rho(\xi) = v\left(\sum_{j=1}^{n} x_j + s - s_{in}\right) = w\left(\sum_{j=1}^{n} y_j + z - s_{in}\right). \tag{2.10}$$

By using the above fact, one can write the variable s in terms of the other variables as follows:

$$s = \frac{\rho(\xi)}{v} - \sum_{j=1}^{n} x_j + s_{in}. \tag{2.11}$$

This is a key step in the approach used in Moreno [20] that reformulates the problem with one species in a planar one. However, since it does not simplify our results, we shall work with all of the variables. In the proof of Proposition 7.4 only, equality (2.11) will be used.

3. The one-impulse and singular arc strategies. From an initial state $\xi = (y,z,w) \in \mathcal{D}$ at time t_0, we define the *immediate one impulse* strategy (that we shall denote *IOI strategy* in the following), which consists in making the following:

1. An impulse of volume $v_{\max} - w$ at t_0. This can be achieved by $r(\tau) = 0$, $u(\tau) = u_{\max}$, for $\tau \in [t_0, t_0 + (v_{\max} - w)/u_{\max}]$.
2. A null control (no feeding) until the concentration $s(\tau)$ reaches s_{out}.

For convenience, we shall denote by $\tilde{y}(\xi)$ and $\tilde{z}(\xi)$ the concentrations obtained with an impulse of volume $v_{\max} - w$ from a state $\xi = (y, z, w) \in \mathcal{D}$:

$$\tilde{y}(\xi) = y\frac{w}{v_{\max}}, \quad \tilde{z}(\xi) = z\frac{w}{v_{\max}} + s_{in}\left(1 - \frac{w}{v_{\max}}\right). \tag{3.1}$$

Notice that, for the particular case $\tilde{z}(\xi) \leq s_{out}$, the first step only is used.

A second strategy considered in this paper is defined as follows. Consider a time t_0, a state $\xi = (y, z, w) \in \mathcal{D}$, a level substrate $\bar{s}$ in $]0, s_{in}[$, and define the quantity

$$s^{\dagger}(\bar{s}, w) = s_{in} - (s_{in} - \max(\bar{s}, s_{out}))\frac{v_{\max}}{w}. \tag{3.2}$$

The *singular arc* strategy on the level $\bar{s}$, denoted by $SA(\bar{s})$, consists of the following steps.

1. *First step:*
   a. If $z > s^{\dagger}(\bar{s}, w)$ and $z < \bar{s}$, make an impulse of volume $w(\bar{s} - z)/(s_{in} - \bar{s})$ at t_0. This can be achieved by $r(\tau) = 0$ and $u(\tau) = u_{\max}$, for $\tau \in [t_0, \bar{t}]$, where $\bar{t} = t_0 + w(\bar{s} - z)/u_{\max}(s_{in} - \bar{s})$ (then s and v jump to $\bar{s}$ and $\bar{v} = w(s_{in} - z)/(s_{in} - \bar{s}) \leq v_{\max}$, respectively).
   b. If $z \geq \bar{s}$ and $z > s^{\dagger}(\bar{s}, w)$, apply a null control (no feeding) until the concentration $s(\cdot)$ reaches the value $\max(\bar{s}, s^{\dagger}(\bar{s}, w))$, i.e., $r(\tau) = 1$, $u(\tau) = 0$ for $\tau \in [t_0, \bar{t}]$, where $\bar{t}$ is such that $s^{t_0,y,z}(\bar{t}) = \max(\bar{s}, s^{\dagger}(\bar{s}, w))$ and $s^{t_0,y,z}(\cdot)$ is the solution of the free dynamics

$$\begin{cases} \dfrac{dx_i}{d\tau} = \mu_i(s)x_i, \quad x_i(t_0) = y_i \quad (i = 1 \cdots n), \\ \dfrac{ds}{d\tau} = -\displaystyle\sum_{j=1}^{n} \mu_j(s)x_j, \quad s(t_0) = z. \end{cases} \tag{3.3}$$

   c. If $z \leq s^{\dagger}(\bar{s}, w)$, make an impulse of volume $v_{\max} - w$ and go the the third step.
2. *Second step:*
   a. If the current state s is equal to $\bar{s}$, make a *singular arc*[1] by taking $r(\tau) = 1$ and a suitable control $u(\cdot)$ ensuring $s(\tau) = \bar{s}$ for any $\tau \in (\bar{t}, \tilde{T}]$, where $\tilde{T}$ is such that $v(\tilde{T}^{+}) = v^{\dagger}(\bar{s})$ and the volume $v^{\dagger}(\bar{s})$ is defined as follows:

$$v^{\dagger}(\bar{s}) = v_{\max} \min\left(1, \frac{s_{in} - s_{out}}{s_{in} - \bar{s}}\right). \tag{3.4}$$

   If $v^{\dagger}(\bar{s}) < v_{\max}$ (or, equivalently, $\bar{s} < s_{out}$), then make an impulse of volume $v_{\max} - v^{\dagger}(\bar{s})$, and the process is finished. Otherwise, go to the third step.
   b. If the current state s is equal to $s^{\dagger}(\bar{s}, w)$, make an impulse of volume $v_{\max} - w$, and the process is finished.
3. *Third step:* Apply $r = 1$ and a null control u until the concentration $s(\cdot)$ reaches s_{out}.

[1]See [4, Part III Chapter 2] for a formal definition.

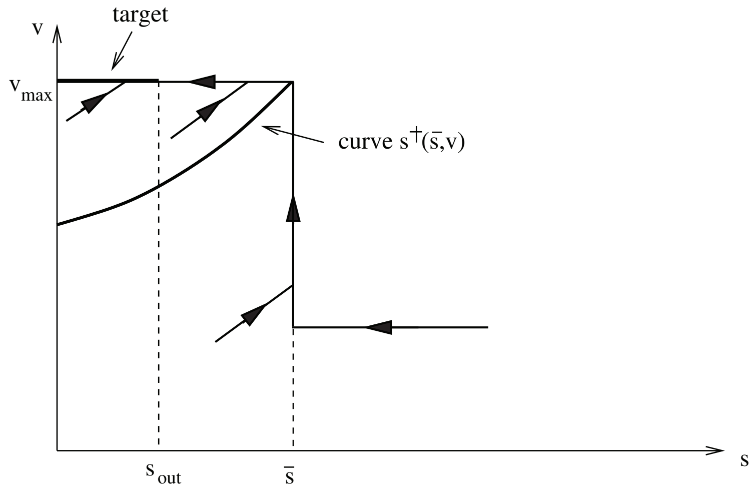

FIG. 3.1. *The* $SA(\bar{s})$ *synthesis when* $\bar{s} > s_{out}$.

Notice that $z \leq s^{\dagger}(\bar{s}, w)$ implies that before reaching the substrate level $\bar{s}$ with an impulse of volume, one reaches the volume $v_{\max}$. When $z > s^{\dagger}(\bar{s}, w)$ and $s^{\dagger}(\bar{s}, w) > \bar{s}$, the variable s reaches the value $s^{\dagger}(\bar{s}, w)$ before $\bar{s}$, and then an impulse drives directly to the target.

Observe also that, in order to apply the singular arc strategy on $\bar{s}$, imposing $ds/d\tau = 0$, the following constraint on the control u must be satisfied:

$$\frac{v}{(s_{in} - \bar{s})} \sum_{j=1}^{n} \mu_j(\bar{s}) x_j = u \leq u_{\max}.$$

Since the maximum level of substrate on which one can apply a singular arc, starting from $\xi \in \mathcal{D}$, is given by $\tilde{z}(\xi)$ defined in (3.1), a sufficient condition, on the initial condition ξ, in order to guarantee the above inequality is to have

$$M \left(\frac{\rho(\xi)}{s_{in} - \tilde{z}(\xi)} + v_{\max} \right) \leq u_{\max}, \tag{3.5}$$

where $\rho(\xi)$ is defined by (2.10) and

$$M = \max_{\substack{s \in [0, s_{in}] \\ j=1,\ldots,n}} \mu_j(s).$$

Indeed, from the definition of $\rho(\xi)$, one has

$$\begin{aligned} \frac{v}{(s_{in} - \bar{s})} \sum_{j=1}^{n} \mu_j(\bar{s}) x_j &\leq M \frac{v}{(s_{in} - \bar{s})} \sum_{j=1}^{n} x_j = M \left(\frac{\rho(\xi)}{s_{in} - \bar{s}} + v \right) \\ &\leq M \left(\frac{\rho(\xi)}{s_{in} - \tilde{z}(\xi)} + v_{\max} \right). \end{aligned}$$

The synthesis of the $SA(\bar{s})$ strategy is depicted on Figures 3.1 and 3.2, depending on the position of $\bar{s}$ relatively to s_{out}.

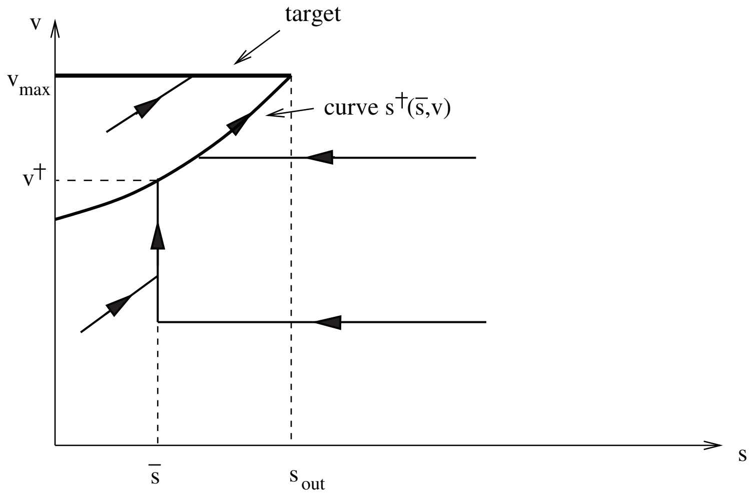

FIG. 3.2. *The* $SA(\bar{s})$ *synthesis when* $\bar{s} < s_{out}$.

Remark 4. An impulse of volume δw at time τ can be achieved by any control law $u(\cdot)$ such that there exists $\delta\tau > 0$ satisfying

$$\int_{\tau}^{\tau+\delta\tau} u(\theta)d\theta = \delta w,$$

with $r(\theta) = 0$ for $\theta \in [\tau, \tau + \delta\tau]$. For the sake of simplicity, we shall systematically take $u(\theta) = u_{\max}$ for $\theta \in [\tau, \tau + \delta w/u_{\max}]$.

4. The cost of the one-impulse strategies. We consider a family of functions $\varphi_c(\cdot)$ defined on $(\mathbb{R}^n_+ \setminus \{0\}) \times \mathbb{R}_+$ and parameterized by $c > 0$:

$$\varphi_c(y, z) = \inf \left\{ t - t_0 \,|\, s^{t_0,y,z}(t) \leq c \right\}, \tag{4.1}$$

where $s^{t_0,y,z}(\cdot)$ is the solution of the free dynamics (3.3). A standard analysis of minimal time problems shows that $\varphi_c(\cdot)$ are Lipschitz-continuous functions and solutions, in the viscosity sense, of the partial differential equation (see, for instance, [2])

$$\sum_{j=1}^{n} (\partial_{y_j}\varphi_c(y, z) - \partial_z\varphi_c(y, z))\mu_j(z)y_j + 1 = 0 \tag{4.2}$$

on the domain $(\mathbb{R}^n_+ \setminus \{0\}) \times (c, +\infty)$, with boundary conditions

$$\varphi_c(.\,, z) = 0 \quad \forall z \in (0, c]. \tag{4.3}$$

The time cost of the IOI strategy can then be simply written in terms of the above functions as follows:

$$T_{IOI}(\xi) = \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)), \tag{4.4}$$

where $\tilde{y}(\xi)$ and $\tilde{z}(\xi)$ are given by (3.1).

Remark 5. Observe that the suboptimal IOI strategy has a finite time cost $T_{IOI}(\xi)$ for a any initial condition ξ in the domain $\mathcal{D}$. Consequently, the optimal value $V(\xi)$ is finite for any ξ in $\mathcal{D}$.

5. The Hamilton–Jacobi characterization. Let us define the *Hamiltonian* as the mapping $H : \mathbb{R}_+^n \times [0, s_{in}] \times [0, v_{\max}] \times [0, u_{\max}] \times [0, 1] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$H(x, s, v, u, r, p, k, q) = r + qu + k\left(\frac{u}{v}(s_{in} - s) - r\sum_{j=1}^{n} x_j \mu_j(s)\right) + \sum_{j=1}^{n} p_j x_j \left(r\mu_j(s) - \frac{u}{v}\right). \tag{5.1}$$

The Hamilton–Jacobi–Bellman equation associated to minimal time problem with dynamics (2.6) is

$$\min_{u\in[0,u_{\max}]} \min_{r\in\{0,1\}} H(y, z, w, u, r, \partial_y V(\xi), \partial_z V(\xi), \partial_w V(\xi)) = 0, \tag{5.2}$$

or equivalently

$$\begin{aligned} &\min\left(0, \sum_{j=1}^{n}(\partial_{y_j} V(\xi) - \partial_z V(\xi))\mu_j(z)y_j + 1\right) \\ &+ \frac{u_{\max}}{w}\min\left(0, -\sum_{j=1}^{n}\partial_{y_j} V(\xi)y_j + \partial_z V(\xi)(s_{in} - z) + \partial_w V(\xi)w\right) = 0, \end{aligned} \tag{5.3}$$

for any $\xi \in \mathcal{D}$, with the boundary condition

$$V(\xi) = 0 \quad \forall \xi \in \mathcal{T}\,. \tag{5.4}$$

Remark 6. Notice that the control variable $r(\cdot)$ does not take values in a convex set. This does not a priori guarantee the existence of an admissible optimal trajectory in $\mathcal{C}$ defined in (2.8). In the following, we prove the existence of optimal trajectories by exhibiting particular strategies for which $r(\cdot)$ takes values 0 or 1.

It is straightforward to check that (5.3) is equivalent to a system of two partial differential inequalities:

$$\Delta_r V(\xi) = \sum_{j=1}^{n}(\partial_{y_j} V(\xi) - \partial_z V(\xi))\mu_j(z)y_j + 1 \geq 0, \tag{5.5}$$

$$\Delta_u V(\xi) = -\sum_{j=1}^{n}\partial_{y_j} V(\xi)y_j + \partial_z V(\xi)(s_{in} - z) + \partial_w V(\xi)w \geq 0, \tag{5.6}$$

independently of the upper bound $u_{\max}$.

Remark 7. Note that the situation when $\Delta_r V$ and $\Delta_u V$ are both strictly positive corresponds to controls $u = 0$ and $r = 0$. Since we consider only controls in $\mathcal{C}$ (defined in (2.8)), we obtain that inequalities (5.5)–(5.6) are equivalent to

$$\min\{\Delta_r V(\xi), \Delta_u V(\xi)\} = 0.$$

As it is well known from the theory of first order Hamilton–Jacobi partial differential equation (p.d.e.) [2], the differentiability of the value function is not guaranteed. Furthermore, in the present case, the uniqueness of the solution of system (5.4)–(5.5)–(5.6) among smooth functions is not guaranteed either (one can easily check that $V \equiv 0$ is always a solution). Nevertheless, one has the following result, providing sufficient conditions for the existence of an optimal trajectory in $\mathcal{C}$ and the smoothness of the value function.

PROPOSITION 5.1. *If there exist*

(a) *a nonnegative continuous function $V(\cdot)$ that fulfills the boundary condition* (5.4) *and such that at any $\xi \in \mathcal{D}$, with $V(\xi) > 0$, $V(\cdot)$ is C^1 and fulfills the partial differential inequalities* (5.5) *and* (5.6);

(b) *two maps $\xi \mapsto u^\star(\xi) \geq 0$, $\xi \mapsto r^\star(\xi) \in \{0,1\}$, with*

$$H(\xi, u^\star(\xi), r^\star(\xi), \nabla V(\xi)) = 0, \quad \forall \xi \in \mathcal{D} \text{ such that (s.t.) } V(\xi) > 0,$$

$$r^\star(\xi) = 0, \quad \forall \xi \in \mathcal{D} \text{ s.t. } V(\xi) = 0,$$

and such that the closed-loop dynamics

$$\tag{5.7} \begin{cases} \dfrac{dx_i}{d\tau} = r^\star(X(\tau))\mu_i(s)x_i - \dfrac{u^\star(X(\tau))}{v}x_i \quad (i = 1 \cdots n), \\ \dfrac{ds}{d\tau} = -r^\star(X(\tau))\displaystyle\sum_{j=1}^{n} \mu_j(s)x_j + \dfrac{u^\star(X(\tau))}{v}(s_{in} - s), \\ \dfrac{dv}{d\tau} = u^\star(X(\tau)), \end{cases}$$

admits an absolutely continuous solution $X(\cdot) = (x(\cdot), s(\cdot), v(\cdot))$ that reaches the target in finite time, for any initial condition $X(t_0) = \xi \in \mathcal{D}$, then $V(\xi)$ is the value function (2.9) *at any $\xi \in \mathcal{D}$ such that the solution of* (5.7) *fulfills $u^\star(X(\tau)) \leq u_{\max}$ for any $\tau \geq t_0$ such that $X(\theta) \notin \mathcal{T}$ whatever is $\theta \in [t_0, \tau)$.*

Proof. Fix an initial condition $\xi \in \mathcal{D}$ at time t_0, and consider admissible controls $(r(\cdot), u(\cdot)) \in \mathcal{C}$ such that the trajectory $X(\cdot) = (x(\cdot), s(\cdot), v(\cdot))$ solution of system (2.6) reaches the target in finite time, say, at time τ_c. Define then the function

$$\mathcal{V}(\tau) = V(X(\tau)),$$

and consider the set $\mathcal{N} = \{\tau \in [t_0, \tau_c] \mid \mathcal{V}(\tau) > 0\}$. Clearly, $\mathcal{V}(\cdot)$ is absolutely continuous on $\mathcal{N}$ and one has

$$\begin{aligned} &\mathcal{V}(\tau_c) - \mathcal{V}(t_0) = \int_{\mathcal{N}} \mathcal{V}'(\tau)d\tau \\ &= \int_{\mathcal{N}} H(x(\tau), s(\tau), v(\tau), u(\tau), r(\tau), \partial_y V(X(\tau)), \partial_z V(X(\tau)), \partial_w V(X(\tau)) - r(\tau)d\tau. \end{aligned}$$

From (5.2), one deduces the inequality

$$\mathcal{V}(\tau_c) - \mathcal{V}(t_0) \geq -\int_{\mathcal{N}} r(\tau)d\tau,$$

and with the boundary condition (5.4),

$$V(\xi) = \mathcal{V}(t_0) \leq \int_{\mathcal{N}} r(\tau)d\tau \leq \int_{t_0}^{\tau_c} r(\tau)d\tau.$$

This last inequality being valid for any admissible controls $(u(\cdot), r(\cdot))$, one deduces that

$$\inf_{u(\cdot),\ r(\cdot)} \left\{ \int_{t_0}^{\tau} r(\theta)d\theta \mid s^{t_0,\xi,u,r}(\tau) \leq s_{out} \ , \ v^{t_0,\xi,u,r}(\tau) = v_{\max} \right\} \geq V(\xi).$$

Consider now the trajectory $X^\star(\cdot)$ solution of (5.7) that reaches the target at time $\tau_c^\star$. The function $\mathcal{V}^\star(\cdot) = V(X^\star(\cdot))$ and the set $\mathcal{N}^\star = \{\tau \in [t_0, \tau_c^\star] \mid \mathcal{V}^\star(\tau) > 0\}$ verify that

$$\mathcal{V}^\star(\tau_c^\star) = \mathcal{V}^\star(t_0) - \int_{\mathcal{N}^\star} r^\star(\tau) d\tau = \mathcal{V}^\star(t_0) - \int_{t_0}^{\tau_c^\star} r^\star(\tau) d\tau.$$

We finally obtain

$$\begin{aligned} V(\xi) = \mathcal{V}^\star(t_0) &= \int_{t_0}^{\tau_c^\star} r^\star(\tau) d\tau \\ &\geq \inf_{u(\cdot),\ r(\cdot)} \left\{ \int_{t_0}^{\tau} r(\theta) d\theta \,|\, s^{t_0,\xi,u,r}(\tau) \leq s_{out}\ ,\ v^{t_0,\xi,u,r}(\tau) = v_{\max} \right\}, \end{aligned}$$

which proves that $V(\cdot)$ is the value function. □

Remark 8. For a function $V(\cdot)$ that fulfills condition (a) of Proposition 5.1 independently of $u_{\max}$, the existence of a pair of admissible feedbacks $u^\star(\cdot)$, $r^\star(\cdot)$ that leads to the target is related to the value of $u_{\max}$.

On the other hand, the above result establish that there exists at most one function V (the value function of our problem) satisfying the conditions of Proposition 5.1. Nevertheless, one cannot conclude the uniqueness of the optimal trajectory.

Remark 9. One can easily check that the function $V \equiv 0$ is a C^1 solution of the Hamilton–Jacobi–Bellman equation (5.3) that fulfills the boundary condition (5.4). But (5.2) imposes to have $r \equiv 0$. Clearly, such controls do not allow one to reach the target, and the conditions of Proposition 5.1 are not fulfilled.

For technicalities, we shall assume in the following that functions $\varphi_c(\cdot)$ defined in (4.1) possess some regularity.

Assumption A0. For any $c > 0$, the function $\varphi_c(\cdot)$ is C^1 on $(\mathbb{R}_+^n \setminus \{0\}) \times (c, +\infty)$.

LEMMA 5.2. *Under Assumption* A0, *at any* $\xi \in \mathcal{D}$ *such that* $T_{IOI}(\xi) > 0$, *one has*

(5.8)
$$\Delta_r T_{IOI}(\xi) = \sum_{j=1}^{n} \left(\partial_{y_j} \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)) - \partial_z \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi))\right) \tilde{y_j}(\mu_j(z) - \mu_j(\tilde{z}(\xi))).$$

Proof. At $\xi = (y, z, w) \in \mathcal{D}$ such that $T_{IOI}(\xi) > 0$, Assumption A0 guarantees that $T_{IOI}(\cdot)$ is C^1. Let us write its partial derivatives as follows:

$$\begin{cases} \partial_{y_j} T_{IOI}(\xi) = \dfrac{w}{v_{\max}} \partial_{y_j} \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)) \qquad (j = 1 \cdots n), \\ \partial_z T_{IOI}(\xi) = \dfrac{w}{v_{\max}} \partial_z \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)), \\ \partial_w T_{IOI}(\xi) = \displaystyle\sum_j \frac{y_j}{v_{\max}} \partial_{y_j} \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)) - \frac{s_{in} - z}{v_{\max}} \partial_z \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)). \end{cases}$$

Then, one has

$$\Delta_r T_{IOI}(\xi) = \sum_j \left(\partial_{y_j}\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi)) - \partial_z\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\right)\mu_j(z)\tilde{y}_j(\xi) + 1. \tag{5.9}$$

Equation (4.2) with $c = s_{out}$ at $(\tilde{y}(\xi),\tilde{z}(\xi))$ provides the equality

$$\sum_j \left(\partial_{y_j}\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi)) - \partial_z\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\right)\mu_j(\tilde{z}(\xi))\tilde{y}_j(\xi) = -1. \tag{5.10}$$

Combining (5.9) and (5.10) gives, finally,

$$\begin{aligned}\Delta_r T_{IOI}(\xi) = \sum_j \Big(&\partial_{y_j}\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\\ &-\partial_z\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\Big)\tilde{y}_j(\xi)(\mu_j(z)-\mu_j(\tilde{z}(\xi))).\end{aligned}$$ ∎

We then obtain the following result concerning the optimality of the IOI strategy for any initial condition.

PROPOSITION 5.3. *Under Assumption* A0, *the IOI strategy is optimal for any* $\xi \in \mathcal{D}$ *if and only if*

$$\Delta_r T_{IOI}(\xi) \geq 0 \quad \forall \xi \in \mathcal{D} \text{ s.t. } T_{IOI}(\xi) > 0. \tag{5.11}$$

Proof. We proceed to show that the function $T_{IOI}(\cdot)$ fulfills conditions of Proposition 5.1.

If $\xi \in \mathcal{T}$, one has $T_{IOI}(\xi) = 0$, thus boundary condition (5.4) is fulfilled. At $\xi \in \mathcal{D}$ such that $T_{IOI}(\xi) > 0$, $T_{IOI}(\cdot)$ is C^1 under assumption A0.

Notice that condition (5.11) is exactly the first partial differential inequality (5.5). The verification of the second partial differential inequality (5.6) is easy:

$$\begin{aligned}\Delta_u T_{IOI}(\xi) = &-\sum_j \partial_{y_j}\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\tilde{y_j} + \partial_z\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\frac{w}{v_{\max}}(s_{in}-z)\\ &+\sum_j \partial_{y_j}\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\tilde{y_j} - \partial_z\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\frac{w}{v_{\max}}(s_{in}-z) = 0.\end{aligned}$$

So, condition (a) of Proposition 5.1 is fulfilled.

Finally, the IOI strategy, as defined in section 3, straightforwardly fulfills condition (b) of Proposition 5.1. ∎

Let us now consider the function

$$\psi(\xi, c) = \varphi_c(y,z) + T_{IOI}(x_c, c, w), \quad \xi \in \mathcal{D},\ c \in (0,z),$$

where $x_c = x^{t_0,y,z}(t_c)$ such that $s^{t_0,y,z}(t_c) = c$, with $(x^{t_0,y,z}(\cdot), s^{t_0,y,z}(\cdot))$ solution of the free dynamics (3.3). Concerning the optimality of the IOI strategy, the study of the function $\psi(\cdot)$ allows us to show that condition (5.11) is also necessary for a given initial condition $\xi \in \mathcal{D}$ such that $T_{IOI}(\xi) > 0$. For this purpose, the next technical lemma will be useful.

LEMMA 5.4. *Under Assumption* A0, *one has*

$$\partial_c\psi(\xi, z) = -\frac{\Delta_r T_{IOI}(\xi)}{\sum_{j=1}^n \mu_j(z)y_j}, \quad \xi \in \mathcal{D} \text{ s.t. } T_{IOI}(\xi) > 0.$$

Proof. From (3.3), one has

$$\frac{\partial x_i(t_c)}{\partial c} = -\frac{\mu_i(c)x_i(t_c)}{\sum_{j=1}^{n}\mu_j(c)x_j(t_c)}, \quad \partial_c\varphi_c(y,z) = -\frac{1}{\sum_{j=1}^{n}\mu_j(c)x_j(t_c)}.$$

Then, one can write

$$\begin{aligned}
\partial_c\psi(\xi,z) &= [\partial_c\varphi_c(y,z)]_{c=z} \\
&\quad + \left[\sum_{j=1}^{n}\partial_{y_j}T_{IOI}(x(t_c),c,w)\frac{\partial x_j(t_c)}{\partial c} + \partial_z T_{IOI}(x(t_c),c,w)\right]_{c=z} \\
&= -\frac{1}{\sum_{j=1}^{n}\mu_j(z)y_j} - \frac{\frac{w}{v_{\max}}\sum_{j=1}^{n}\partial_{y_j}\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi))\mu_j(z)y_j}{\sum_{j=1}^{n}\mu_j(z)y_j} \\
&\quad + \frac{w}{v_{\max}}\partial_z\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi)) \\
&= -\frac{1+\sum_{j=1}^{n}(\partial_{y_j}\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi)) - \partial_z\varphi_{s_{out}}(\tilde{y}(\xi),\tilde{z}(\xi)))\mu_j(z)\tilde{y}_j(\xi)}{\sum_{j=1}^{n}\mu_j(z)y_j}.
\end{aligned}$$

Using the property (4.2) for $c = s_{out}$ at $(\tilde{y}(\xi),\tilde{z}(\xi))$, and the expression (5.8) given by Lemma 5.2, finally gives

$$\partial_c\psi(\xi,c) = -\frac{\Delta_r T_{IOI}(\xi)}{\sum_{j=1}^{n}\mu_j(z)y_j}. \qquad \square$$

Proposition 5.3 states that if (5.11) is not satisfied, then there exists an initial condition $\xi \in \mathcal{D}$ for which the IOI strategy is not optimal. The following proposition characterizes some initial conditions for when this occurs.

PROPOSITION 5.5. *At states $\xi \in \mathcal{D}$ such that $T_{IOI}(\xi) > 0$ and $\Delta_r T_{IOI}(\xi) < 0$, the IOI strategy cannot be optimal.*

Proof. When $T_{IOI}(\xi) > 0$ and $\Delta_r T_{IOI}(\xi) < 0$ at $\xi \in \mathcal{D}$, Lemma 5.4 gives the existence of $c^\star < z$ such that $\psi(\xi,c^\star) < \psi(\xi,z) = T_{IOI}(\xi)$. Consequently, there is a strategy (consisting in applying a null control until the time $t_{c^\star}$ such that $s(t_{c^\star}) = c^\star$ and then the IOI strategy) which has a better cost than the IOI strategy. $\square$

6. Derivation from the minimum principle. In this section we apply the Pontryagin's minimum principle (PMP) (see [4, 25]) to the minimal time problem with dynamics (2.6).

The PMP states that when $(x,s,v,u,r)(\cdot)$ is a solution of the minimal time problem associated to the system (2.6), then there exists an n-dimensional *multiplier* $p(\cdot)$

and scalar multipliers $q(\cdot)$ and $k(\cdot)$ such that

$$
\left\{
\begin{aligned}
\frac{dp_i}{d\tau} &= p_i u/v - r(p_i - k)\mu_i(s), \quad p_i(T) = 0,\\
\frac{dk}{d\tau} &= -r\sum_{j=1}^{n}(p_j - k)x_j\mu_j'(s) + ku/v, \quad k(T) = 1,\\
\frac{dq}{d\tau} &= \frac{u}{v^2}\left(k(s_{in} - s) - \sum_{j=1}^{n} p_j x_j\right),
\end{aligned}
\right.
\tag{6.1}
$$

where T is the optimal terminal time. In addition, the Hamiltonian (defined in (5.1))

$$(u, r) \longrightarrow H(x(\tau), s(\tau), v(\tau), u, r, p(\tau), k(\tau), q(\tau))$$

is minimized in $u(\tau)$ and $r(\tau)$, at any $\tau \in [t_0, T]$.

Define the auxiliary variables $\tilde{p}_i = p_i - k$, who play an important role in what follows.

First, we observe that the dynamics of $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n)$ can be written as follows:

$$\frac{d\tilde{p}}{d\tau} = A(\tau)\tilde{p}, \qquad \tilde{p}_i(T) = -1, \tag{6.2}$$

where $A(\tau)$ is an $n \times n$ time dependent matrix. Consequently, one has $\tilde{p}(\tau) \neq 0$ for any $\tau \in [t_0, T]$.

On another hand, one has

$$\operatorname*{argmin}_{u,r} H(x, s, v, u, r, p, k, q) = \operatorname*{argmin}_{u,r} u\phi_u(x, s, v, p, k, q) + r\phi_r(x, s, p, k), \tag{6.3}$$

where

$$
\left\{
\begin{aligned}
\phi_u(x, s, v, p, k, q) &= q + \frac{k}{v}(s_{in} - s) - \frac{1}{v}\sum_{j=1}^{n} p_j x_j,\\
\phi_r(x, s, p, k) &= 1 + \sum_{j=1}^{n}(p_j - k)\mu_j(s)x_j = 1 + \sum_{j=1}^{n}\tilde{p}_j\mu_j(s)x_j.
\end{aligned}
\right.
\tag{6.4}
$$

If we derive with respect to the fictitious time τ, we obtain

$$
\left\{
\begin{aligned}
\frac{d\phi_u}{d\tau} &= -r\frac{(s_{in} - s)}{v}\langle\tilde{p}, m\rangle,\\
\frac{d\phi_r}{d\tau} &= u\frac{(s_{in} - s)}{v}\langle\tilde{p}, m\rangle,
\end{aligned}
\right.
\tag{6.5}
$$

where $m = m(\tau)$ is given by

$$m(\tau) = \begin{pmatrix} \mu_1'(s(\tau))x_1(\tau) \\ \vdots \\ \mu_n'(s(\tau))x_n(\tau) \end{pmatrix} \tag{6.6}$$

and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product in $\mathbb{R}^n$. Finally, it is straightforward to check that (see (6.2))

$$\frac{d}{d\tau}\langle\tilde{p}, m\rangle = \left\langle\tilde{p}, A^\top m + \frac{dm}{d\tau}\right\rangle. \tag{6.7}$$

The next result links the optimal value function $V(\cdot)$ to Pontryagin's multipliers $(p,q,k)(\cdot)$.

PROPOSITION 6.1. *Let $V(\cdot)$ be the optimal value function defined in* (2.2). *Consider an initial vector $\xi=(y,z,w)\in\mathcal{D}$ and denote by $(p,q,k)(\cdot)$ the corresponding Pontryagin's multipliers. If the value function V is C^1 at ξ, it holds that*

$$\Delta_r V(\xi)=\phi_r(x,s,p,k) \quad \text{and} \quad \Delta_u V(\xi)=v\phi_u(x,s,p,k,q), \tag{6.8}$$

where $\Delta_r V(\xi)$ and $\Delta_u V(\xi)$ are defined by (5.5) *and* (5.6), *respectively.*

Proof. See Theorem 12.5.1 in [31]. ∎

We end this section by introducing our second assumption and a lemma whose proof is direct.

Assumption A1. The functions $\mu_i(\cdot)$ are nondecreasing.

LEMMA 6.2. *Under Assumption* A1, *the following assertions hold:*

i. *The matrix $A(\tau)$ has nonnegative off-diagonal terms, i.e., the dynamical system* (6.2) *is cooperative (see* [28]*);*
ii. *The vector $m(\cdot)$, defined in* (6.6), *lies in $\mathbb{R}^n_+$.*

7. The one-species case. For this case, it is straightforward to check that, for any $c>0$, the partial differential equation (4.2) with boundary condition (4.3) admits the (unique) nonnegative continuous solution that is C^1 on $(\mathbb{R}_+\setminus\{0\})\times(c,+\infty)$, given by the expression

$$\varphi_c(y,z)=\left|\begin{array}{ll}\displaystyle\int_c^z \frac{d\zeta}{\mu(\zeta)(y+z-\zeta)} & \text{for } z>c,\\ 0 & \text{for } z\le c.\end{array}\right. \tag{7.1}$$

Hence, Assumption A0 is fulfilled. We give a technical lemma that will be useful in the following.

LEMMA 7.1. *Let $z>c>0$. If $\mu(\cdot)$ is nonincreasing on $[c,z]$, then the following inequalities are satisfied:*

$$\partial_z\varphi_c(y,z)\ge\frac{1}{\mu(z)(y+z-c)}, \tag{7.2}$$

$$\partial_z\varphi_c(y,z)\le\frac{1}{\mu(c)(y+z-c)}+\frac{1}{\mu(z)y}-\frac{1}{\mu(c)y}. \tag{7.3}$$

Proof. Notice first that the partial derivative of function φ_c given in (7.1) verifies

$$\partial_z\varphi_c(y,z)=\frac{1}{\mu(z)y}-\int_c^z\frac{d\zeta}{\mu(\zeta)(y+z-\zeta)^2}.$$

Since $\mu(\cdot)$ is nonincreasing, one can write

$$\partial_z\varphi_c(y,z)\ge\frac{1}{\mu(z)y}-\int_c^z\frac{d\zeta}{\mu(z)(y+z-\zeta)^2}=\frac{1}{\mu(z)(y+z-c)},$$

and

$$\begin{aligned}\partial_z\varphi_c(y,z)&\le\frac{1}{\mu(z)y}-\int_c^z\frac{d\zeta}{\mu(c)(y+z-\zeta)^2}\\&=\frac{1}{\mu(z)y}+\frac{1}{\mu(c)(y+z-c)}-\frac{1}{\mu(c)y}.\end{aligned}$$

∎

7.1. Increasing growth functions.

PROPOSITION 7.2. *Under Assumption* A1, *the IOI strategy is optimal for any initial condition ξ in $\mathcal{D}$, and the value function is*

$$V(\xi) = \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)), \tag{7.4}$$

where $\varphi_{s_{out}}(\cdot)$ is given by (7.1) *and $(\tilde{y}(\xi), \tilde{z}(\xi))$ by* (3.1).

Proof. For $(\tilde{y}(\xi), \tilde{z}(\xi))$ such that $\tilde{z}(\xi) > s_{out}$, the map $\xi \mapsto \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi))$ is C^1 at ξ, and from (4.2), one has

$$(\partial_y \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)) - \partial_z \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)))\mu(\tilde{z}(\xi))\tilde{y}(\xi) = -1.$$

Then, condition (5.11) of Proposition 5.3 simply becomes $\mu(\tilde{z}(\xi)) - \mu(z) \geq 0$, which is fulfilled when $\mu(\cdot)$ is nondecreasing and $\tilde{z} \geq z$. ∎

Remark 10. This proposition extends to impulse controls a result obtained by Moreno [20] for measurable control (with a different technique based on Green's theorem). It states that, for a monotonic growth functions μ, the *one bang* control

$$F = \begin{cases} F_{\max} & \text{if } v < v_{\max}, \\ 0 & \text{if } v = v_{\max}, \end{cases}$$

is optimal.

It is clear that there is no uniqueness of optimal control laws (see Remarks 2 and 4). Eventually, we could have another control $(u(\cdot), r(\cdot))$ that implies a strategy different to the *IOI* one, with the same value function defined in (7.4). The following proposition shows that the IOI strategy is, in fact, the unique admissible optimal one. This result is obtained using PMP.

PROPOSITION 7.3. *Under Assumption* A1, *one has that, for any initial condition ξ in $\mathcal{D}$, the IOI strategy is the unique optimal control law.*

Proof. From the PMP, there exist multipliers p and k solutions of system (6.1), which, in this case ($n = 1$), satisfies $\langle \tilde{p}, m \rangle = \tilde{p}\mu'(s)x < 0$ for m defined by (6.6) and $\tilde{p} = p - k$. Then, from (6.5), one has that $\frac{d\phi_r}{d\tau} \leq 0$ and $\frac{d\phi_u}{d\tau} \geq 0$ along all of the optimal trajectories. Also, relations (6.8), (5.5), and (5.6) imply that $\phi_u \geq 0$ and $\phi_r \geq 0$.

Since the admissible controls (u, r) are in $\mathcal{C}$ defined by (2.8) (see Remark 2) and the states of the system (2.6) must reach the target $\mathcal{T}$, the only possibilities for ϕ_u and ϕ_r are as follows:

i. $\phi_u = 0$ at the beginning and then $\phi_u > 0$;
ii. $\phi_r > 0$ at the beginning and then $\phi_r = 0$.

Indeed, the admissible control set $\mathcal{C}$ allows us to consider only configurations such that $\phi_u \cdot \phi_r = 0$, and $u \neq 0$ or $r \neq 0$. This, together with (6.5), discards the choices $\phi_u > 0$ or $\phi_r = 0$, at the beginning.

On the other hand, (6.5) implies also that $u \neq 0$ until the time when ϕ_u switches from $\phi_u = 0$ to $\phi_u > 0$. This time coincides with the time when $v = v_{\max}$ and, consequently, also with the time when ϕ_r switches from $\phi_r > 0$ to $\phi_r = 0$ (because u becomes null at such time). Therefore, one has the following:

i. $u = u_{\max}$ until v reaches $v_{\max}$ and then $u = 0$;
ii. $r = 0$ until $v = v_{\max}$ and then $u = 0$ and $r = 1$ until the concentration s reaches s_{out}.

This proves the optimality of the IOI strategy, and, by construction, we have uniqueness. ∎

7.2. Nonmonotonic growth functions with one maximum. In this section we consider a continuously differentiable growth function $\mu(\cdot)$, which is nonmonotonic and attains a unique isolated maximum point $s^* \in (0, s_{in})$. More precisely, this growth function satisfies $\mu'(s) > 0$ for all $s \in [0, s^*)$, $\mu'(s) < 0$ for all $s > s^*$, and $\mu'(s^*) = 0$.

One instance of such functions is typically the Haldane law, given by the expression

$$\mu(s) = \frac{\bar{\mu} s}{K + s + s^2/R},$$

where K is the affinity constant and R is the inhibition constant. This kind of growth function occurs in bioprocesses where the substrate is a toxic substance and, for big concentrations, inhibits the activity of the microorganisms [29].

The following proposition solves our minimal time problem for this type of growth function. This solution has been previously obtained in [20] for the class of measurable and bounded controls, and under the assumption $s^* > s_{out}$. Furthermore, we give the expression of the value function $V(\cdot)$. For convenience, we define the number

$$s^{\diamondsuit} = \max(s^*, s_{out}).$$

PROPOSITION 7.4. *For any initial condition $\xi = (y, z, w) \in \mathcal{D}$ that satisfies* (3.5)*, the $SA(s^*)$ strategy (defined in section* 3*) is optimal. Furthermore, the value function at ξ is given by the expression*

$$V(\xi) = \begin{cases} \varphi_{s^*}(y,z) + \gamma(\xi) + \varphi_{s_{out}}\left(\tilde{y}(\xi) + \tilde{z}(\xi) - s^{\diamondsuit}, s^{\diamondsuit}\right), & \text{for } z > s^* \geq s^{\dagger}(s^*, w), \\ \gamma\left(y\dfrac{s_{in} - s^*}{s_{in} - z}, s^*, w\dfrac{s_{in} - z}{s_{in} - s^*}\right) + \varphi_{s_{out}}\left(\tilde{y}(\xi) + \tilde{z}(\xi) - s^{\diamondsuit}, s^{\diamondsuit}\right), & \text{for } z \leq s^* \text{ and } z > s^{\dagger}(s^*, w), \\ \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)), & \text{for } z \leq s^{\dagger}(s^*, w), \\ \varphi_{s^{\dagger}(s^*,w)}(y,z) & \text{for } z > s^{\dagger}(s^*, w) > s^*, \end{cases} \tag{7.5}$$

where $\varphi_{\cdot}(\cdot)$ is given by (7.1)*, $\tilde{y}(\cdot)$ and $\tilde{z}(\cdot)$ by* (3.1)*, $s^{\dagger}(\cdot)$ by* (3.2)*, and*

$$\gamma(\xi) = \frac{1}{\mu(s^*)} \log\left(\frac{w(y + z - s_{in}) + v_{\max}\left(s_{in} - s^{\diamondsuit}\right)}{w(y + z - s^*)}\right). \tag{7.6}$$

Proof. First, observe that if the initial condition ξ is in the target, i.e., $z \leq s_{out}$ and $w = v_{\max}$, then $s^{\dagger}(s^*, w) = s^{\diamondsuit} \geq s_{out} \geq z$ and $\tilde{z}(\xi) = z$ and, therefore it corresponds to the third case in the definition of V obtaining $\varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi)) = 0$, and hence the boundary condition (5.4) is satisfied. We prove now that $V(\cdot)$ is C^1 and fulfills the Hamilton–Jacobi inequalities (5.5) and (5.6), concluding the result from Proposition 5.1 because one can easily check that $V(\xi)$ is the cost associated to the $SA(s^*)$ strategy from initial condition ξ.

For $z > s^*$ and $s^* \geq s^\dagger(s^*, w)$, $V(\cdot)$ is C^1 and its partial derivatives are

(7.7)
$$\begin{aligned}
\partial_y V(\xi) &= \partial_y \varphi_{s^*}(y,z) + \frac{1}{\mu(s^*)}\left(\frac{w}{w(y+z-s_{in})+(s_{in}-s^\diamondsuit)v_{\max}} - \frac{1}{y+z-s^*}\right) \\
&\qquad + \partial_y \varphi_{s_{out}}\left(\tilde{y}(\xi)+\tilde{z}(\xi)-s^\diamondsuit, s^\diamondsuit\right)\frac{w}{v_{\max}}, \\
\partial_z V(\xi) &= \partial_z \varphi_{s^*}(y,z) + \frac{1}{\mu(s^*)}\left(\frac{w}{w(y+z-s_{in})+(s_{in}-s^\diamondsuit)v_{\max}} - \frac{1}{y+z-s^*}\right) \\
&\qquad + \partial_y \varphi_{s_{out}}\left(\tilde{y}(\xi)+\tilde{z}(\xi)-s^\diamondsuit, s^\diamondsuit\right)\frac{w}{v_{\max}}, \\
\partial_w V(\xi) &= \frac{1}{\mu(s^*)}\left(\frac{y+z-s_{in}}{w(y+z-s_{in})+(s_{in}-s^\diamondsuit)v_{\max}} - \frac{1}{w}\right) \\
&\qquad + \partial_y \varphi_{s_{out}}\left(\tilde{y}(\xi)+\tilde{z}(\xi)-s^\diamondsuit, s^\diamondsuit\right)\frac{y+z-s_{in}}{v_{\max}}.
\end{aligned}$$

Then, one has straightforwardly

$$\begin{aligned}
\Delta_r V(\xi) &= \left(\partial_y \varphi_{s^*}(y,z) - \partial_z \varphi_{s^*}(y,z)\right)\mu(z) y + 1, \\
\Delta_u V(\xi) &= -\partial_y \varphi_{s^*}(y,z) y + \partial_z \varphi_{s^*}(y,z)(s_{in}-z) + \frac{1}{\mu(s^*)}\left(\frac{y+z-s_{in}}{y+z-s^*} - 1\right).
\end{aligned}$$

Using the property (4.2) fulfilled by the function φ_{s^*}, one obtains

$$\begin{aligned}
\Delta_r V(\xi) &= 0, \\
\Delta_u V(\xi) &= (s_{in}-s^*)\left(\partial_z \varphi_{s^*}(y,z) - \frac{1}{\mu(s^*)(y+z-s^*)}\right) \\
&\qquad + (y+z-s^*)\left(\frac{1}{\mu(z)(y+z-s^*)} - \partial_z \varphi_{s^*}(y,z)\right).
\end{aligned}$$

Consequently, inequality (5.5) is fulfilled. For the second inequality (5.6), we distinguish two cases:

i. $y+z-s_{in} \leq 0$. Since $\mu(z) \leq \mu(s^*)$, one can write the inequality

$$\Delta_u V(\xi) \geq (y+z-s_{in})\left(\frac{1}{\mu(s^*)(y+z-s^*)} - \partial_z \varphi_{s^*}(y,z)\right),$$

and with inequality (7.2) given by Lemma 7.1 with $c = s^*$, one deduces that

$$\Delta_u V(\xi) \geq \frac{y+z-s_{in}}{y+z-s^*}\left(\frac{1}{\mu(s^*)} - \frac{1}{\mu(z)}\right) \geq 0.$$

ii. $y+z-s_{in} > 0$. With inequality (7.3) given by Lemma 7.1 with $c = s^*$, one can write

$$\begin{aligned}
\Delta_u V(\xi) &= (s_{in}-y-z)\partial_z \varphi_{s^*}(y,z) + \frac{1}{\mu(z)} - \frac{s_{in}-s^*}{\mu(s^*)(y+z-s^*)} \\
&\geq (s_{in}-y-z)\left(\frac{1}{\mu(z)y} + \frac{1}{\mu(s^*)(y+z-s^*)} - \frac{1}{\mu(s^*)y}\right) + \frac{1}{\mu(z)} \\
&\qquad - \frac{1}{\mu(s^*)} - \frac{s_{in}-y-z}{\mu(s^*)(y+z-s^*)} \\
&= \frac{s_{in}-z}{y}\left(\frac{1}{\mu(z)} - \frac{1}{\mu(s^*)}\right) \geq 0.
\end{aligned}$$

For $z < s^*$ and $z > s^\dagger(s^*, w)$, $V(\cdot)$ is C^1 and its partial derivatives are

(7.8)
$$\begin{aligned}
\partial_y V(\xi) &= \frac{1}{\mu(s^*)}\left(\frac{w}{w(y+z-s_{in})+(s_{in}-s^\diamondsuit)v_{\max}} - \frac{1}{y}\right) \\
&\qquad + \partial_y \varphi_{s_{out}}\left(\tilde{y}(\xi)+\tilde{z}(\xi)-s^\diamondsuit, s^\diamondsuit\right)\frac{w}{v_{\max}}, \\
\partial_z V(\xi) &= \frac{1}{\mu(s^*)}\frac{w}{w(y+z-s_{in})+(s_{in}-s^\diamondsuit)v_{\max}} \\
&\qquad + \partial_y \varphi_{s_{out}}\left(\tilde{y}(\xi)+\tilde{z}(\xi)-s^\diamondsuit, s^\diamondsuit\right)\frac{w}{v_{\max}}, \\
\partial_w V(\xi) &= \frac{1}{\mu(s^*)}\left(\frac{y+z-s_{in}}{w(y+z-s_{in})+(s_{in}-s^\diamondsuit)v_{\max}} - \frac{1}{w}\right) \\
&\qquad + \partial_y \varphi_{s_{out}}\left(\tilde{y}(\xi)+\tilde{z}(\xi)-s^\diamondsuit, s^\diamondsuit\right)\frac{y+z-s_{in}}{v_{\max}}.
\end{aligned}$$

One has clearly, from (7.1),

$$\partial_y \varphi_{s^*}(y, s^*) = 0 \quad \text{and} \quad \partial_z \varphi_{s^*}(y, s^*) = \frac{1}{\mu(s^*)y},$$

which implies, from expressions (7.7) and (7.8),

$$\lim_{z\to s^*+} \nabla V(\xi) = \lim_{z\to s^*-} \nabla V(\xi).$$

Consequently, $V(\cdot)$ is C^1 at points ξ such that $z = s^*$. From the expressions (7.8), one can straightforward check the following equalities:

$$\Delta_r V(\xi) = 1 - \frac{\mu(s^*)}{\mu(z)} \geq 0 \quad \text{and} \quad \Delta_u V(\xi) = 0.$$

Consider now points ξ such that $z < s^\dagger(s^*, w)$ and $V(\xi) > 0$. At such points, $V(\cdot)$ is C^1 and its partial derivatives are

$$\begin{cases}
\partial_y V(\xi) = \partial_y \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi))\dfrac{w}{v_{\max}}, \\
\partial_z V(\xi) = \partial_z \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi))\dfrac{w}{v_{\max}}, \\
\partial_w V(\xi) = \partial_y \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi))\dfrac{y}{v_{\max}} - \partial_z \varphi_{s_{out}}(\tilde{y}(\xi), \tilde{z}(\xi))\dfrac{s_{in}-z}{v_{\max}}.
\end{cases}$$

Notice that when $z = s^\dagger(s^*, w)$, with $s^\dagger(s^*, w) > s^*$, one has $V(\xi) = 0$. One can easily check that $V(\cdot)$ is also C^1 at points ξ such that $z = s^\dagger(s^*, w)$ and $V(\xi) > 0$, because, at such points, one has $\tilde{z}(\xi) = s^\diamondsuit = s^*$, and furthermore, function $\varphi_{s_{out}}(\cdot)$ fulfills the property (from (4.2))

$$\partial_z \varphi_{s_{out}}(\tilde{y}(\xi), s^*) = \partial_y \varphi_{s_{out}}(\tilde{y}(\xi), s^*) + \frac{1}{\mu(s^*)\tilde{y}(\xi)}.$$

Notice that one has also $V(\xi) = T_{IOI}(\xi)$. Since $\Delta_u T_{IOI}(\xi) = 0$ (see the proof of Proposition 5.3), the function $V(\cdot)$ is a solution of the Hamilton–Jacobi equation

(5.3) at ξ when the single condition $\Delta_r T_{IOI}(\xi) \geq 0$ is fulfilled. Here, this condition simply becomes $\mu(\tilde{z}(\xi)) - \mu(z) = \mu(s^*) - \mu(z) \geq 0$, which is fulfilled because s^* is the maximum of the function μ.

Finally, we consider situations for which $z > s^\dagger(s^*, w) > s^*$. This case occurs only when $s^* < s_{out}$. At such points $\xi = (y, z, w)$, the function $V(\cdot)$ is C^1 and its partial derivatives are

$$\begin{cases} \partial_y V(\xi) = \partial_y \varphi_{s^\dagger(s^*,w)}(y, z), \\ \partial_z V(\xi) = \partial_z \varphi_{s^\dagger(s^*,w)}(y, z), \\ \partial_w V(\xi) = -\dfrac{(s_{in} - s_{out})v_{\max}}{\mu(s^\dagger(s^*, w))(y + z - s^\dagger(s^*, w))w^2}. \end{cases}$$

One can easily check that the first inequality $\Delta_r V(\xi) \geq 0$ is fulfilled. For the second one, let us write

$$\begin{aligned} \Delta_u V(\xi) &= -\partial_y \varphi_{s^\dagger(s^*,w)}(y, z)y + \partial_z \varphi_{s^\dagger(s^*,w)}(y, z)(s_{in} - z) \\ &\qquad - \frac{(s_{in} - s_{out})v_{\max}}{\mu(s^\dagger(s^*, w))(y + z - s^\dagger(s^*, w))w^2} \\ &= \frac{1}{\mu(z)} + \partial_z \varphi_{s^\dagger(s^*,w)}(y, z)(s_{in} - z - y) \\ &\qquad - \frac{(s_{in} - s_{out})v_{\max}}{\mu(s^\dagger(s^*, w))(y + z - s^\dagger(s^*, w))w^2} \end{aligned} \tag{7.9}$$

from the expression (4.2). We distinguish now two cases.

i. $s_{in} - z - y \geq 0$. Expressions (7.9) and (7.2) given by Lemma 7.1 with $c = s^\dagger(s^*, w)$ give together

$$\Delta_u V(\xi) \geq \frac{s_{in} - s^\dagger(s^*, w)}{y + z - s^\dagger(s^*, w)} \left(\frac{1}{\mu(z)} - \frac{1}{\mu(s^\dagger(s^*, w))} \right) \geq 0.$$

ii. $s_{in} - z - y < 0$. Gathering expressions (7.9) and (7.3) given by Lemma 7.1 with $c = s^\dagger(s^*, w)$ leads to the following inequality:

$$\Delta_u V(\xi) \geq \frac{s_{in} - z}{y} \left(\frac{1}{\mu(z)} - \frac{1}{\mu(s^\dagger(s^*, w))} \right) \geq 0. \qquad \square$$

Remark 11. For initial conditions $\xi = (y, z, w)$, with $z \geq s^* > s^\dagger(s^*, w)$, the value of $\gamma(\xi)$, defined by (7.6), represents the time spent on the singular arc $s = s^*$ from volume w up to volume $v^\dagger(\bar{s})$ defined in (3.4). Indeed, at time $t_1 = \varphi_{s^*}(y, z)$, one has $s(t_1) = s^*$, $v(t_1) = w$, and $x(t_1) = y + z - s^*$ (see the invariant $\rho(\xi)$ defined in (2.10)). Then, the suitable control in order to keep $s = s^*$ is obtained from the equation $\dfrac{ds}{d\tau} = 0$, that is,

$$u = u_s(v) = \frac{\mu(s^*)}{s_{in} - s^*}\rho(\xi) + v\mu(s^*).$$

This implies that the $v(\cdot)$ is the solution of the ordinary differential equation

$$\frac{dv}{d\tau} = u_s(v) = \frac{\mu(s^*)}{s_{in} - s^*}w(y + z - s_{in}) + \mu(s^*)v$$

up to time t_2 such that $v(t_2) = v^\dagger$, that can be solved analytically, leading to $t_2 - t_1 = \gamma(\xi)$.

Remark 12. Proposition 7.4 extends to impulse controls the result obtained by Moreno [20] for measurable controls, based on a Green's theorem argumentation.

Moreover, notice that when $s^* < s_{out}$, the $SA(s^*)$ strategy imposes a final impulse before reaching the target. Such situations were not considered in [20].

Remark 13. Proposition 7.4 establishes that the time associated to the singular arc strategy is the optimal value function, but we could have that there exists another optimal control different to this strategy. Nevertheless, one can prove uniqueness (analogously as in Proposition 7.3) using the PMP. We will not develop this approach here because it is very similar to the one used in Proposition 7.3 and in the proof of Theorem 8.2 below.

8. The two-species case. We first consider functions $\mu_i(\cdot)$ that are C^2 and such that μ_1'/μ_2' is a strictly monotonic function. Without loss of generality, we assume that μ_1'/μ_2' is strictly decreasing, which is equivalent to the following condition.

Assumption A2. $\mu_1'(s)\mu_2''(s) > \mu_1''(s)\mu_2'(s)$ for any $s \in (0, s_{in}]$.

Remark 14. Assumption A2 is fulfilled for nonproportional Monod growth functions. That is, for growth functions

$$\mu_i(s) = \frac{\mu_{\max,i} s}{K_i + s},$$

Assumption A2 holds when $K_1 < K_2$.

LEMMA 8.1. *Under Assumption* A2, *a singular arc*[2] I *is characterized by* $\frac{ds}{d\tau} = 0$ *on* I.

Proof. Consider p and k solutions of PMP-system (6.1) and m defined in (6.6). Define the auxiliary variable $\tilde{p} = p - k$. Recall that the property $\tilde{p}(\tau) \neq 0$, for any time τ, follows from (6.2).

If I is a singular arc, it follows that the first derivatives of ϕ_u and ϕ_r are null on I. Since controls u and r are not simultaneously null, it is equivalent to write $\langle \tilde{p}, m \rangle = 0$ on I (via (6.5)). Differentiating this last equation w.r.t. τ and using expression (6.7), it holds that

$$\langle \tilde{p}, m \rangle = 0 \quad \text{and} \quad \left\langle \tilde{p}, A^\top m + \frac{dm}{d\tau} \right\rangle = 0 \quad \text{on } I. \tag{8.1}$$

Since $\tilde{p}$ is always nonnull and has dimension 2, equalities (8.1) are satisfied if m and $A^\top m + \frac{dm}{d\tau}$ are linearly dependent on I. We easily verify that, under A2, this is equivalent to $\frac{ds}{d\tau} = 0$ on I. Indeed, a simple computation leads to

$$\left\| m \times \left(A^\top m + \frac{dm}{d\tau} \right) \right\| = \left| \frac{ds}{d\tau} \right| \; x_1 x_2 \; (\mu_1'(s)\mu_2''(s) - \mu_1''(s)\mu_2'(s)).$$

Vectors $A^\top m + \frac{dm}{d\tau}$ are linearly dependent exactly when their cross product (in $\mathbb{R}^3$) is null, which holds if and only if $\frac{ds}{d\tau} = 0$. We have thus proved that if I is a singular arc, then $\frac{ds}{d\tau} = 0$ on I.

[2]See [4, Part III Chapter 2] for an exact definition. In our case, a singular arc consists of an open interval of time I, where $\phi_u = \phi_r = 0$, and then no information on controls u and r can be obtained directly from (6.3).

Reciprocally, suppose that $\frac{ds}{d\tau} = 0$ on I. The above arguments imply that m and $A^\top m + \frac{dm}{d\tau}$ are linearly dependent on I, obtaining from (6.7) that $\frac{d\langle \tilde{p}, m\rangle}{d\tau} = \lambda(\tau)\langle \tilde{p}, m\rangle$ on I, for a real-valued continuous map $\lambda : I \to \mathbb{R}$.

This is equivalent to saying that $\langle \tilde{p}, m\rangle = Ce^{\int_0^\tau \lambda(t)dt}$ on I for a real constant C and a real-valued continuous map $\lambda : I \to \mathbb{R}$. Moreover, by replacing $\frac{ds}{d\tau} = 0$ in (2.6), it necessarily requires strictly positive controls u and r.

Suppose first that $C = 0$. Then (6.5) implies that ϕ_u and ϕ_r are both constant on I. Since u and r are strictly positive, this holds only if $\phi_u = \phi_r = 0$ on I, i.e., I is a singular arc.

Now, suppose that $C \neq 0$. One obtains from (6.5) that ϕ_u and ϕ_r are both strictly monotonic on I. This, together with inequalities (5.5)–(5.6), implies that ϕ_u and ϕ_r are both strictly positive on I. But, since u and r are also strictly positive, it contradicts (6.3). We thus conclude that $C = 0$, and hence I is a singular arc. ∎

Consider now the next assumption on growth functions $\mu_i(\cdot)$.

Assumption A3. For any $s_1,\ s_2\ \in [s_{out}, s_{in}]$, one has

$$s_2 \geq s_1 \ \Rightarrow\ \mu_2(s_2)\mu_1(s_1) \geq \mu_1(s_2)\mu_2(s_1). \tag{8.2}$$

Remark 15. Assumption A3 is fulfilled for growth functions constant or linear on $[s_{out}, s_{in}]$. For Monod functions

$$\mu_i(s) = \frac{\mu_{\max,i} s}{K_i + s},$$

condition (8.2) is exactly fulfilled when $K_1 \leq K_2$.

THEOREM 8.2. *Assume that Assumptions* A0-A1–A2–A3 *are fulfilled. Then, for any initial condition in $\mathcal{D}$ that satisfies* (3.5)*, the optimal solution of the minimal time problem associated to dynamics* (2.6) *consists in either the IOI strategy or the SA*(s^*) *strategy for some* $s^* \in]s_{out}, s_{in}[$.

To prove this theorem, we need the three following technical lemmas.

LEMMA 8.3. *Under Assumptions* A1 *and* A3, *one has*

$$\tilde{p}_1 \geq 0 \ \ \textit{and} \ \ \phi_r = 0 \ \Rightarrow\ \langle \tilde{p}, m\rangle \leq 0. \tag{8.3}$$

Proof of Lemma 8.3. Observe that Assumption A3 implies that $\mu_1'\mu_2 - \mu_1\mu_2' \leq 0$. If $\phi_r = 0$, then

$$\tilde{p}_2 = \frac{-(1 + \tilde{p}_1\mu_1 x_1)}{\mu_2 x_2},$$

and then

$$\langle \tilde{p}, m\rangle = \frac{\tilde{p}_1}{\mu_2}(\mu_1'\mu_2 - \mu_1\mu_2')x_1 - \frac{\mu_2'}{\mu_2},$$

which proves the desired result. ∎

LEMMA 8.4. *Under Assumptions* A1 *and* A2, *for* $\tilde{p}(\tau) \in E(\tau) = \{\tilde{p} = (\tilde{p}_1, \tilde{p}_2) \mid \langle \tilde{p}, m(\tau)\rangle = 0\}$, *one has*

$$\operatorname{sgn}\left(\frac{d}{d\tau}\langle \tilde{p}, m\rangle\right) = -\operatorname{sgn}\left(\frac{ds}{d\tau}\tilde{p}_1\right). \tag{8.4}$$

Proof of Lemma 8.4. Let $\tilde{p}(\tau) = (\tilde{p}_1, \tilde{p}_2)$ be in $E(\tau)$, that is, $\tilde{p}_2 = -\tilde{p}_1\mu_1'(s)x_1/\mu_2'(s)x_2$. It is straightforward to check that the property

$$\frac{d}{d\tau}\langle \tilde{p}, m\rangle = \left\langle A^\top m + \frac{dm}{d\tau}, \tilde{p}\right\rangle = \frac{ds}{d\tau}\tilde{p}_1\left[\mu_1'' - \frac{\mu_1'\mu_2''}{\mu_2'}\right]x_1$$

is fulfilled, from which (8.4) is deduced (recalling A2). $\square$

LEMMA 8.5. *If, for some interval of time $[\tau_-, \tau_+]$, one has $\phi_r = \phi_u = 0$, then*

(a) *if $\tilde{p}_1 > 0$, ϕ_r and ϕ_u remain equal to zero for all $\tau \geq \tau_+$;*

(b) *if $\tilde{p}_1 < 0$, either $\langle \tilde{p}, m\rangle \geq 0$ for all $\tau \geq \tau_+$ or $\langle \tilde{p}, m\rangle \leq 0$ for all $\tau \geq \tau_+$.*

Proof of Lemma 8.5. Notice that if we have $\phi_r = \phi_u = 0$ in some interval of time, necessarily, by (6.5), $\langle \tilde{p}, m\rangle = 0$ on this interval. Since $(\tilde{p}_1, \tilde{p}_2) \neq (0, 0)$ for all τ and the vector m lies in $]0, +\infty[\times]0, +\infty[$ (under Assumption 1), we deduce that $\tilde{p}_1 \neq 0$, and therefore it does not change its sign in this interval.

Suppose now that $\bar{\tau} = \sup\{\tau \mid \langle \tilde{p}, m\rangle = 0\} < +\infty$. If

$$\exists\ \bar{\delta} > 0 \quad \text{such that} \quad \forall\ \delta \in]0, \bar{\delta}], \quad \text{one has} \quad \langle \tilde{p}(\bar{\tau} + \delta), m(\bar{\tau} + \delta)\rangle > 0, \tag{8.5}$$

since ϕ_r and ϕ_u have to be nonnegative, necessarily, according to (6.5), the control r must be zero, $u = u_{\max}$, and therefore $\frac{ds}{d\tau} > 0$ until $\langle \tilde{p}, m\rangle$ changes its sign.

(a) If $\tilde{p}_1 > 0$ on $[\tau_-, \tau_+]$, let us prove that $\langle \tilde{p}, m\rangle$ does not become positive. If it occurs, we have (8.5), and, in such a case, $\frac{ds}{d\tau} > 0$. Nevertheless, by (8.4), we obtain $\frac{d}{d\tau}\langle \tilde{p}, m\rangle \leq 0$ at $\bar{\tau}$, which contradicts (8.5).
With similar arguments, one can prove that $\langle \tilde{p}, m\rangle$ does not become negative, and hence $\bar{\tau} = +\infty$.

(b) For the case $\tilde{p}_1 < 0$, if we have (8.5), then $\frac{ds}{d\tau} > 0$ until $\langle \tilde{p}, m\rangle$ changes its sign. This change will never happen, because, from (8.4), one has $\frac{d}{d\tau}\langle \tilde{p}, m\rangle \geq 0$ on the set $E(\tau) = \{\tilde{p} = (\tilde{p}_1, \tilde{p}_2) \mid \langle \tilde{p}, m(\tau)\rangle = 0\}$, and therefore, $\langle \tilde{p}, m\rangle$ remains nonnegative.
Analogous arguments allow us to prove that if there exists $\bar{\delta} > 0$ such that for all $\delta \in]0, \bar{\delta}]$ one has $\langle \tilde{p}(\bar{\tau} + \delta), m(\bar{\tau} + \delta)\rangle < 0$, then $\langle \tilde{p}, m\rangle$ remains nonpositive. $\square$

As a corollary of Lemma 8.5, one has that if the optimal strategy includes a singular arc ($\phi_r = \phi_u = 0$ during an interval), then it must occur with $\tilde{p}_1 < 0$. Indeed, if $\tilde{p}_1 > 0$ and $\phi_r = \phi_u = 0$ during an interval, the last equalities will remain for every larger time. This situation is not allowed because $\tilde{p}$ must be equal to $(-1, -1)$ at the final time.

On the other hand, if the optimal strategy includes a singular arc (with $\tilde{p}_1 < 0$), after this process, necessarily, $\langle \tilde{p}, m\rangle$ must be negative. In fact, if $\langle \tilde{p}, m\rangle$ is positive, it does not change its sign, and then $\tilde{p}$ will never be equal to $(-1, -1)$.

Thus, if a singular arc occurs, the volume at the end of this process must be equal to $v_{\max}$ because, as $\langle \tilde{p}, m\rangle$ will remain nonpositive and ϕ_r and ϕ_u have to be nonnegative, the control u is equal to zero and $r = 1$ for the rest of time, and then the process of filling the tank has necessarily finished.

As a last consequence of Lemma 8.5, we obtain that in the case of a singular arc, which is equivalent to keep the level of substrate s constant (see Lemma 8.1), this level has to be greater than s_{out}. Indeed, if it is not the case, we have that all of the processes finish at the end of the singular arc because $v = v_{\max}$ and s_{out} is greater than the current substrate level, and hence the target has been reached. This cannot occur, because, at the end, the vector $\tilde{p}$ must be equal to $(-1, -1)$.

Proof of Theorem 8.2. As in the proof of Proposition 7.3, the positivity of ϕ_u and ϕ_r plays a crucial role (cf. (6.8), (5.5), and (5.6)).

Moreover, the considered admissible control set $\mathcal{C}$ (see Remark 2) tell us that only optimal controls such that $u \neq 0$ or $r \neq 0$ are considered. And, therefore, ϕ_u and ϕ_r cannot be both strictly positive.

Recall that, under Assumption A1, the matrix A of system (6.2) is cooperative and the vector m, defined by (6.6), is always in $]0,+\infty[\times]0,+\infty[$. Then, since $\tilde{p} = (p_1 - k, p_2 - k)$ is equal to $(-1,-1)$ at the final time T, we deduce that $\tilde{p} \notin \mathbb{R}_+^2$ at any time τ, and moreover, once $\tilde{p}$ reaches the negative octant $\mathbb{R}_-^2$, it remains there until time T. Thus, thanks to Lemma 8.4, the study of the sign of $\tilde{p}_1$ will be a key issue in the proof. Indeed, the arguments above imply that either $\tilde{p}_1$ remains always negative, or it is positive at initial time and then it becomes negative until the final time T.

Hence, our proof splits into the following two cases.

Case 1: $\tilde{p}_1(t_0) > 0$. Let us first discard the following case:

(a) $v < v_{\max}$, $\tilde{p}_1 > 0$, $\langle \tilde{p}, m\rangle < 0$, and $\phi_r = 0$.

In this situation, one has necessarily $u = 0$ and $r = 1$ in order to keep ϕ_r nonnegative. Thus, s decreases, ϕ_u increases, and there exists a time τ such that $\langle \tilde{p}, m\rangle = 0$ at τ and $\langle \tilde{p}, m\rangle > 0$ for larger time (because $\frac{d}{d\tau}\langle \tilde{p}, m\rangle > 0$, cf. (8.4)). Since $\phi_r = 0$, we obtain a contradiction with Lemma 8.3.

Thus, if $v < v_{\max}$, $\tilde{p}_1 > 0$, and $\langle \tilde{p}, m\rangle < 0$, then $\phi_r > 0$ and $\phi_u = 0$. In such a case, one has $r = 0$ and $u = u_{\max}$, which implies that s increases and ϕ_r decreases until a time such that $\phi_r = 0$ (in order to reach the target). Since a singular arc is not possible with $\tilde{p}_1 > 0$ (see Lemma 8.5), we discard the case $\langle \tilde{p}, m\rangle = \frac{d}{d\tau}\langle \tilde{p}, m\rangle = 0$. On the other hand, as the case (a) above is not possible and $\frac{d}{d\tau}\langle \tilde{p}, m\rangle \leq 0$ (on the set $E(\tau)$), the equality $v = v_{\max}$ has to be fulfilled when $\phi_r = 0$. For larger times, since ϕ_r must be nonnegative, one has $u = 0$, $r = 1$, and then the obtained trajectory is exactly synthesized by the IOI strategy.

If $v < v_{\max}$, $\tilde{p}_1 > 0$, and $\langle \tilde{p}, m\rangle > 0$, from Lemma 8.3, the unique possibility is to have $\phi_r > 0$, and consequently $\phi_u = 0$. In such a case, one has $r = 0$ and $u = u_{\max}$. Hence, s and ϕ_r increase. Then, there exists necessarily a time such that $\langle \tilde{p}, m\rangle = 0$, in order to reach the target. Note that, for larger time, $\langle \tilde{p}, m\rangle < 0$ holds due to $\frac{d}{d\tau}\langle \tilde{p}, m\rangle < 0$. In this situation, ϕ_r decreases until $\phi_r = 0$, and from Lemma 8.3, it has to coincide with the time at which $v = v_{\max}$. After, since ϕ_r must be nonnegative, one has $u = 0$ and $r = 1$. The obtained trajectory is again synthesized by the IOI strategy.

Hence, we have proved that when $\tilde{p}_1(t_0)$ is positive, the IOI strategy is optimal.

Case 2: $\tilde{p}_1(t_0) < 0$. Recall, from the above discussion, that $\tilde{p}_1$ remains always negative. We now proceed to discard the following cases:

(b) $v < v_{\max}$, $\tilde{p}_1 < 0$, $\phi_u > 0$, and $\langle \tilde{p}, m\rangle < 0$.

This case implies that $u = 0$ and $r = 1$. Thus, s decreases, and, by (6.5), ϕ_u increases. This together with (8.4) imply that the sign of $\frac{d}{d\tau}\langle \tilde{p}, m\rangle$ is negative (on the set $E(\tau)$ defined in Lemma 8.4). Consequently, $\langle \tilde{p}, m\rangle$ remains always negative, and ϕ_u always increases. Then the target cannot be reached because the tank is never fulfilled.

(c) $v < v_{\max}$, $\tilde{p}_1 < 0$, $\phi_u = 0$, and $\langle \tilde{p}, m\rangle > 0$.

In this case, one necessarily obtains $r = 0$ and $u = u_{\max}$ (in order to keep ϕ_u nonnegative). Hence, s and ϕ_r increase (see (6.5)). This together with (8.4) imply that the sign of $\frac{d}{d\tau}\langle \tilde{p}, m\rangle$ is positive (on $E(\tau)$). Consequently, $\langle \tilde{p}, m\rangle$ remains always positive, which is a contradiction with $\tilde{p} = (-1,-1)$ at the final time.

Hence, if $v < v_{\max}$, $\tilde{p}_1 < 0$, and $\langle \tilde{p}, m\rangle > 0$ necessarily $\phi_u > 0$ and $\phi_r = 0$, then one has $u = 0$ and $r = 1$ implying that s and ϕ_u decrease until $\langle \tilde{p}, m\rangle = 0$. Equation (8.4) allows us to say that the sign of $\frac{d}{d\tau}\langle \tilde{p}, m\rangle$ is nonpositive, and hence $\langle \tilde{p}, m\rangle \leq 0$ for larger times. If at time such that $\langle \tilde{p}, m\rangle = 0$, we have $\phi_u > 0$ immediately after one has $\langle \tilde{p}, m\rangle < 0$. Then $u = 0$ (in order to keep ϕ_r nonnegative), and consequently ϕ_u remains always positive, and the tank will not be fulfilled, which discard this case. Thus, necessarily φ_u becomes zero when $\langle \tilde{p}, m\rangle = 0$. If $\langle \tilde{p}, m\rangle$ remains equal to zero for an interval of time, this corresponds to a singular arc. If not, that is, $\langle \tilde{p}, m\rangle < 0$ immediately after, this situation implies that $u = 0$ and $r = 1$ onwards, which is impossible because the tank will never be filled.

Finally, we study the remaining case $v < v_{\max}$, $\tilde{p}_1 < 0$, $\phi_u = 0$, and $\langle \tilde{p}, m\rangle < 0$. One has necessarily $r = 0$ and $u = u_{\max}$, in order to keep ϕ_u nonnegative. Therefore, s increases and ϕ_r decreases (see (6.5)) until a time τ^* when one of the following three cases occur:

- case $\phi_r > 0$ and $\langle \tilde{p}, m\rangle = 0$. This implies that $u = u_{\max}$ and $r = 0$, and consequently s increases. This together with (8.4) implies that the sign of $\frac{d}{d\tau}\langle \tilde{p}, m\rangle$ is positive. Consequently, ϕ_r will always remain positive, which cannot allow one to reach the target. This case is thus discarded.
- case $\phi_r = 0$ and $\langle \tilde{p}, m\rangle < 0$. This implies that $u = 0$ and $r = 1$, and therefore (by (6.5)) ϕ_u becomes positive. Since (b) of Case 2 above has been discarded, it necessarily follows that v reaches $v_{\max}$ at the same time τ^*. The optimal trajectory is synthesized by the IOI strategy.
- case $\phi_r = 0$ and $\langle \tilde{p}, m\rangle = 0$. Due to equality $\phi_u = 0$ holding at the same time τ^*, this configuration corresponds to a singular arc.

Thus, we have proved that if $(u, r)(\cdot) \in \mathcal{C}$ is optimal, then it corresponds to an IOI strategy or to the singular arc strategy for a level $s^* \geq s_{out}$. We finish concluding that a singular arc cannot be applied on a substrate level greater that s_{in} because the domain $\mathcal{D} = (\mathbb{R}_+^n \setminus \{0\})\times]0, s_{in}]\times]0, v_{\max}[$ is invariant. ◻

Remark 16. The value of s^* depends on the initial condition and cannot be, in general, explicitly determined as in the case with one nonmonotonic species (see Proposition 7.4).

We give now an example that shows that an $SA(\cdot)$ strategy can be better than an IOI strategy.

Example 1. We consider functions $\mu_1(\cdot)$, $\mu_2(\cdot)$ that fulfill Assumptions A1, A2, and A3, but not A4:

$$\mu_1(s) = s^2, \quad \mu_2(s) = 5\sqrt{s}$$

for the values $s_{out} = 0.1$ and $s_{in} = 5$ (see Figure 8.1).

We compare the strategies IOI and $SA(s^*)$, where s^* minimizes the cost of the $SA(\bar{s})$ strategy for $\bar{s} \in (s_{out}, s_{in})$. For $v_{\max} = 10$ and initial conditions with $y_1 = 1$, $z = 3$, and $w = 1$, we have computed numerically s^* for different values of y_2. Results are reported in Table 8.1.

This example shows that, in the presence of a small population of a species more efficient for small substrate concentrations, the singular arc strategy may be better than the IOI one.

We focus now on sufficient conditions for which the IOI strategy is always optimal. We first consider functions $\mu_i(\cdot)$ such that their graphs do not cross away from 0 (without loss of generality, one can assume that μ_2 is above μ_1).

Assumption A4. $\mu_2(s) \geq \mu_1(s) \quad \forall s \in (0, s_{in}]$.

Then, the functions $\varphi_c(\cdot)$ possess the following properties.

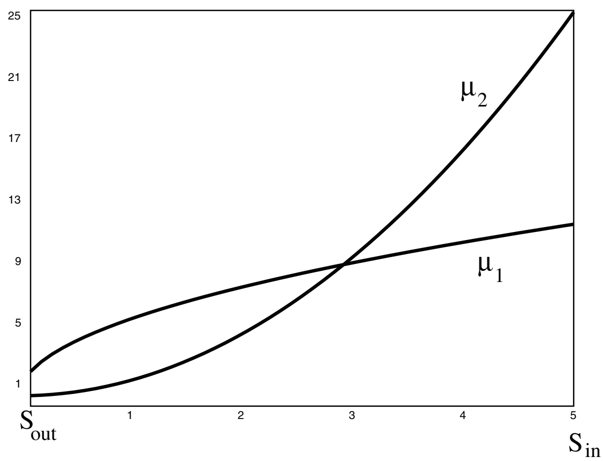

FIG. 8.1. *Graphs of the two growth functions.*

TABLE 8.1

y_2	$T(IOI)$	s^*	$T(SA(s^*))$	Gain
0	2.320765	*not reached*	–	–
10^{-4}	2.126756	1.678	1.973976	7%
10^{-3}	1.700494	1.97	1.515415	11%
10^{-2}	1.186231	2.46	1.101530	7%
0.05	0.867009	3.05	0.842255	3%
0.1	0.746446	3.34	0.739046	1%
0.5	0.522361	*not reached*	–	–

LEMMA 8.6. *Under Assumptions* A0, A1, *and* A4, *for any* $c \in (0, s_{in})$, $(y, z) \in (\mathbb{R}_+^2 \setminus \{0\}) \times (c, s_{in}]$, *one has*

$$\begin{array}{ll} \text{i.} & \partial_z \varphi_c(y,z) \geq \partial_{y_2}\varphi_c(y,z), \\ \text{ii.} & \partial_{y_1}\varphi_c(y,z) \geq \partial_{y_2}\varphi_c(y,z), \\ \text{iii.} & \partial_{y_2}\varphi_c(y,z) \leq 0. \end{array}$$

Proof. Notice that the dynamics (3.3) possesses the property that $M = x_1 + x_2 + s$ is constant. Then, fix a value M, and consider the reduced system

$$\begin{cases} \dfrac{dx_1}{d\tau} = f_1(x_1, s) = \mu_1(s)x_1, \\ \dfrac{ds}{d\tau} = f_2(x_1, s, M) = -\mu_1(s)x_1 - \mu_2(s)(M - s - x_1). \end{cases} \tag{8.6}$$

The Jacobian matrix $\mathcal{J}_M$ of this two-dimensional vector field has the structure

$$\mathcal{J}_M(x_1, s) = \begin{pmatrix} \star & \mu_1'(s)x_1 \\ \mu_2(s) - \mu_1(s) & \star \end{pmatrix},$$

which has nonnegative off-diagonal terms for any fixed M. So, the dynamical system (8.6) is cooperative (see [28]).

i. Consider two sets of initial conditions for system (3.3):

$$(y_1^-, y_2^-, z^-) = (y_1, y_2 + \delta, z) \quad \text{and} \quad (y_1^+, y_2^+, z^+) = (y_1, y_2, z + \delta),$$

where δ is a positive (small) number. We check that these two initial conditions have the same invariant $x_1+x_2+s=M+\delta$, and, from the cooperative property of system (8.6) with $M+\delta$, one has $s^+(t) \geq s^-(t)$ for any $t \geq 0$. Thus $\varphi_c(y_1,y_2,z+\delta) \geq \varphi_c(y_1,y_2+\delta,z)$, or equivalently

$$\frac{\varphi_c(y_1,y_2,z+\delta)-\varphi_c(y_1,y_2,z)}{\delta} \geq \frac{\varphi_c(y_1,y_2+\delta,z)-\varphi_c(y_1,y_2,z)}{\delta},$$

and letting δ take arbitrary small values, we obtain

$$\partial_z\varphi_c(y,z) \geq \partial_{y_2}\varphi_c(y,z).$$

ii. Similarly, we consider the sets of initial conditions

$$(y_1^-,y_2^-,z^-)=(y_1,y_2+\delta,z) \quad \text{and} \quad \left(y_1^+,y_2^+,z^+\right)=(y_1+\delta,y_2,z)$$

and obtain, when δ tends toward zero,

$$\partial_{y_1}\varphi_c(y,z) \geq \partial_{y_2}\varphi_c(y,z).$$

iii. We consider the sets of initial conditions

$$(y_1^-,y_2^-,z^-)=(y_1,y_2+\delta,z) \quad \text{and} \quad \left(y_1^+,y_2^+,z^+\right)=(y_1,y_2,z),$$

with nonnegative δ. The first initial condition leads to the invariant $x_1+x_2+s=M+\delta$, while the second one has $x_1+x_2+s=M$ as an invariant. Dynamics (8.6) is such that $f_2(x_1,s,M+\delta) \leq f_2(x_1,s,M)$, and, by the cooperative property, we conclude that $\varphi_c(y_1,y_2+\delta,z) \leq \varphi_c(y_1,y_2,z)$ or equivalently

$$\partial_{y_2}\varphi_c(y,z) \leq 0. \qquad \square$$

Remark 17. From expression (3.1), one can notice that inequality

$$\tilde{z} \geq z \tag{8.7}$$

is always fulfilled. Thus, under assumptions A0, A1, and A4, if the function $\varphi_{s_{out}}(\cdot)$ is such that

$$\partial_z\varphi_{s_{out}}(y,z) \geq \partial_{y_1}\varphi_{s_{out}}(y,z), \quad \forall (y,z) \in (\mathbb{R}_+^2 \setminus \{0\}) \times (s_{out},s_{in}], \tag{8.8}$$

then, along with point i. of Lemma 8.6, inequality (8.7), and Lemma 5.2, we deduce immediately that condition (5.11) of Proposition 5.3 is fulfilled. Unfortunately, condition (8.8) is rarely met, even for simple growth rates. In Figure 8.2, we plot iso-values of the function

$$\psi_c(y)=\varphi_c(y,M-y_1-y_2)$$

(computed numerically) for $\mu_1(s)=s$, $\mu_2(s)=5s$, $M=10$, and $c=1$. We see that $\partial_{y_2}\psi_c(\cdot)$ is everywhere nonpositive, but the sign of $\partial_{y_1}\psi_c(\cdot)$ can change. Notice that the partial derivatives of $\psi_c(\cdot)$ are linked to the partial derivatives of $\varphi_c(\cdot)$ as follows:

$$\partial_{y_j}\psi_c(y)=\partial_{y_j}\varphi_c(y,z)-\partial_z\varphi_c(y,z), \qquad (j=1,2),$$

with $z=M-y_1-y_2$, and we conclude that condition (8.8) is not fulfilled.

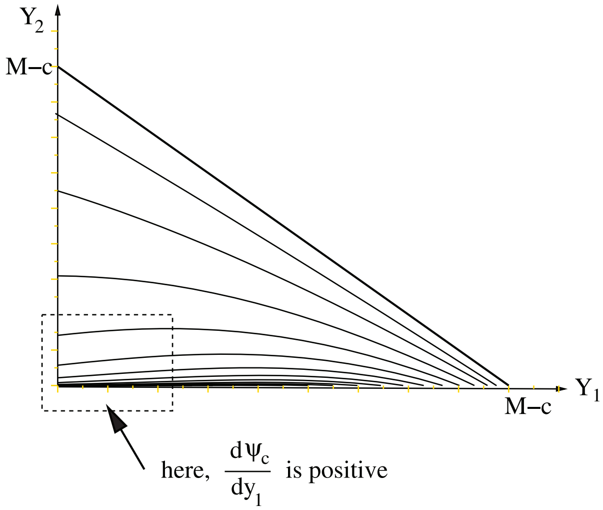

FIG. 8.2. *Iso-values of the function* $\psi_c(\cdot)$.

The following proposition imposes conditions on growth functions $\mu_i(\cdot)$ that guarantee the optimality of the IOI strategy.

PROPOSITION 8.7. *Under Assumptions* A0, A3, *and* A4, *the IOI strategy is optimal for any initial condition in* $\mathcal{D}$.

Proof. Consider $\xi \in \mathcal{D}$ such that $T_{IOI}(\xi) > 0$. Let us write $\Delta_r T_{IOI}(\xi)$ given in (5.8) as follows:

$$\Delta_r T_{IOI}(\xi) = \sum_{j=1}^{2} \left(\partial_{y_j} \varphi_{s_{out}}(\tilde{y}, \tilde{z}) - \partial_z \varphi_{s_{out}}(\tilde{y}, \tilde{z})\right) \mu_j(\tilde{z}) \tilde{y_j} \frac{\mu_j(z) - \mu_j(\tilde{z})}{\mu_j(\tilde{z})}.$$

Recall that $\tilde{z} \geq z$ (8.7) and $\mu_i(\cdot)$ are nondecreasing (Assumption A1). Then, by Assumption A3, one has

$$\frac{\mu_2(z) - \mu_2(\tilde{z})}{\mu_2(\tilde{z})} \leq \frac{\mu_1(z) - \mu_1(\tilde{z})}{\mu_1(\tilde{z})} \leq 0.$$

Lemma 8.6 gives the inequality

$$\partial_{y_2} \varphi_{s_{out}}(\tilde{y}, \tilde{z}) - \partial_z \varphi_{s_{out}}(\tilde{y}, \tilde{z}) \leq 0,$$

and consequently, from equation (4.2), we obtain

$$\begin{aligned} \Delta_r T_{IOI}(\xi) &\geq \sum_{j=1}^{2} \left(\partial_{y_j} \varphi_{s_{out}}(\tilde{y}, \tilde{z}) - \partial_z \varphi_{s_{out}}(\tilde{y}, \tilde{z})\right) \mu_j(\tilde{z}) \tilde{y_j} \frac{\mu_1(z) - \mu_1(\tilde{z})}{\mu_1(\tilde{z})} \\ &= -\frac{\mu_1(z) - \mu_1(\tilde{z})}{\mu_1(\tilde{z})} \geq 0 \end{aligned}$$

and conclude by Proposition 5.3. ∎

9. Conclusion. In this work, we have analyzed the minimal time problem for fed-batch reactors with several species, for which impulse controls are allowed. We have shown that even when all of the growth functions are monotonic, the most rapid approach strategy is not necessarily optimal. In certain situations, it is better to follow a singular arc instead of applying an impulse, a departure from the optimal strategy in the one-species case. We believe that this result holds important implications for biotechnological applications.

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