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Equilibrium problems involving the Lorentz cone

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Abstract We study a general equilibrium model formulated as a smooth system of equations coupled with complementarity conditions relative to the n-dimensional Lorentz cone. For the purpose of analysis, as well as for the design of algorithms, we exploit the fact that the Lorentz cone is representable as a cone of squares in a suitable Euclidean Jordan algebra.

Keywords Lorentz cone · Euclidean Jordan algebra · Complementarity problem · System of nonlinear equations

Mathematics Subject Classification 90C33

1 Introduction

A large variety of equilibrium models arising in applications can be formulated as a nonlinear system of equations coupled with complementarity conditions relative to a closed convex cone. The cone under consideration is usually the nonnegative orthant of $\mathbb{R}^n$, but sometimes one needs to consider the Lorentz cone

$$\mathbb{L}_n := \left\{ x \in \mathbb{R}^n : x_n \geq \left[x_1^2 + \cdots + x_{n-1}^2 \right]^{1/2} \right\},$$

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also known as the second-order cone or the ice-cream cone. Geometrically speaking, $\mathbb{L}_n$ is a revolution cone whose central axis is the half-line generated by the n-dimensional vector $e_n = (\mathbf{0}_{n-1}, 1)$. Here and in the sequel the symbol $\mathbf{0}_d$ refers to the d-dimensional zero vector. This paper addresses the issue of analyzing and solving a nonlinear system

$$f(x, y, \lambda) = \mathbf{0}_n \tag{1}$$

$$g(x) = \mathbf{0}_m, \tag{2}$$

where $x, y \in \mathbb{R}^n$ are unknown vectors linked by means of complementary conditions of the type

$$\mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n. \tag{3}$$

Here $\perp$ indicates orthogonality with respect to the usual inner product of $\mathbb{R}^n$ and $\succeq$ is the order relation induced by the Lorentz cone.

The entries of x and y are called primal and dual variables, respectively. The Eq. (2) reflects a normalization condition and/or a structural constraint imposed on the primal variables. For achieving more generality in the modeling, the Eq. (1) includes an unknown vector of exogenous variables $\lambda_1, \ldots, \lambda_p$. Exogenous variables are not restricted by complementarity, so they are different in nature from the pair of primal and dual variables.

Although (3) cannot be written in a componentwise manner as in the classical theory of complementarity problems, such inconvenience is largely compensated by the rich algebraic structure of the Lorentz cone. We shall elaborate on this issue in Sect. 3.

The purpose of this work is to study and design algorithms for solving the Lorentz Equilibrium Model

$$\mathrm{LEM}(f, g) \quad \begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ f(x, y, \lambda) = \mathbf{0}_n \\ g(x) = \mathbf{0}_m, \end{cases} \tag{4}$$

as well as the particular version

$$\mathrm{LEM}(f) \quad \begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ f(x, y, \lambda) = \mathbf{0}_n \end{cases} \tag{5}$$

in which the normalization or constraint function g is missing. The name attributed to this kind of equilibrium model emphasizes the role played by the Lorentz cone. Some comments on hypotheses are appropriate:

- For avoiding a situation of overdetermination in (4) one assumes that $p \geq m \geq 1$. The choice $p = 0$, which by convention means that f is free of exogenous variables, is tolerated in the particular model (5).
- For simplicity in the presentation one supposes that $f : \mathbb{R}^{2n+p} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions (i.e., continuously differentiable). Much of our analysis extends however to a nonsmooth setting.

The next definition distinguishes between different sorts of solutions to a LEM. The interior and boundary of the Lorentz cone are given respectively by

$$\mathrm{int}(\mathbb{L}_n) = \left\{x \in \mathbb{R}^n : x_n > \left[x_1^2 + \cdots + x_{n-1}^2\right]^{1/2}\right\},$$

$$\mathrm{bd}(\mathbb{L}_n) = \left\{x \in \mathbb{R}^n : x_n = \left[x_1^2 + \cdots + x_{n-1}^2\right]^{1/2}\right\}.$$

Definition 1.1 A solution (x^*, y^*, λ^*) to a LEM is called trivial if $x^* = \mathbf{0}_n$, otherwise it is nontrivial. Furthermore,

(i) A nontrivial solution is of interior type (respectively, boundary type) if x^* belongs to the interior (respectively, boundary) of $\mathbb{L}_n$.
(ii) An interior type solution is central if x^* lies on the central axis of $\mathbb{L}_n$, otherwise it is eccentric.

Remark 1.2 As a consequence of Moreau's decomposition theorem [18], the component y^* of an interior type solution is necessarily equal to $\mathbf{0}_n$. By contrast, for a boundary type solution one may have $y^* = \mathbf{0}_n$ or not.

The remaining part of this introductory section reviews some interesting examples of equilibrium problems that fit into the framework of a LEM.

1.1 LCP relative to the Lorentz cone

Let $\mathbb{M}_n$ be the space of square matrices of order n. The Linear Complementarity Problem relative to the Lorentz cone reads as follows:

$$\text{(LCP)} \quad \begin{cases} \text{given a matrix } M \in \mathbb{M}_n \text{ and a vector } q \in \mathbb{R}^n, \\ \text{find a vector } x \in \mathbb{R}^n \text{ such that} \\ \mathbf{0}_n \preceq x \perp (Mx + q) \succeq \mathbf{0}_n. \end{cases} \tag{6}$$

In this example the Eq. (1) is free of exogenous variables and the Eq. (2) does not show up, because neither normalization nor structural constraints are imposed on x. So, one needs to solve the particular LEM

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ Mx + q - y = \mathbf{0}_n. \end{cases}$$

LCP's relative to the Lorentz cone are considered in [10,17]. Existence results for (6) can be derived by adapting [8, Theorem 12] or various abstract theorems stated in [6, Chapter 2].

As a mathematical curiosity we mention a variant of (6) in which the matrix M depends on a certain number of exogenous variables and the solution x is sought for instance on a unit sphere:

$$\begin{cases} \text{given a smooth function } M : \mathbb{R}^p \to \mathbb{M}_n \text{ and a vector } q \in \mathbb{R}^n, \\ \text{find } \lambda \in \mathbb{R}^p \text{ and a unit vector } x \in \mathbb{R}^n \text{ such that} \\ \mathbf{0}_n \preceq x \perp (M(\lambda)x + q) \succeq \mathbf{0}_n. \end{cases}$$

The later equilibrium problem corresponds to the particular LEM

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ M(\lambda)x + q - y = \mathbf{0}_n \\ \|x\|^2 - 1 = 0. \end{cases} \tag{7}$$

1.2 AVE relative to the Lorentz cone

Given a vector $b \in \mathbb{R}^n$ and matrices $A, B \in \mathbb{M}_n$, one must solve the Absolute Value Equation

$$Az + B|z| = b. \tag{8}$$

The absolute value of $z \in \mathbb{R}^n$ is taken with respect to the Lorentz cone, i.e., $|z| = z_+ + z_-$ with z_+ denoting the projection of z onto $\mathbb{L}_n$ and $z_- = z_+ - z$. AVE's relative to the nonnegative orthant are studied in [13–15,21,23] and in many other places. AVE's relative to the Lorentz cone are considered in [9]. As pointed out in [9, Theorem 1.1], the Eq. (8) can be reformulated as

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ (B + A)x + (B - A)y - b = \mathbf{0}_n, \end{cases}$$

i.e., as a particular instance of the model (5).

1.3 Second-order cone programming

Consider a conic optimization problem of the form

$$\inf\{c(x) : x \succeq \mathbf{0}_n,\ Ax = b\}, \tag{9}$$

where $c : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function, $b \in \mathbb{R}^m$, and A is a real matrix of size $m \times n$. This type of optimization problem has been widely studied in the last decade, specially when c is a linear function (cf. [2]). If one introduces a Lagrange multiplier vector $\lambda \in \mathbb{R}^m$ for the equality constraint $Ax = b$ and a Karush-Kuhn-Tucker multiplier vector $y \in \mathbb{R}^n$ for the nonnegativity constraint $x \succeq \mathbf{0}_n$, then the optimality conditions for (9) can be expressed in the form

$$\begin{aligned} x \succeq \mathbf{0}_n,\ Ax = b &\quad \text{primal feasibility} \\ y \succeq \mathbf{0}_n &\quad \text{dual feasibility} \\ \langle x, y\rangle = 0 &\quad \text{complementarity slackness} \\ \nabla c(x) - A^T\lambda - y = \mathbf{0}_n &\quad \text{stationarity,} \end{aligned}$$

where the superscript "T" indicates transposition. One gets in this way a LEM

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ \nabla c(x) - A^T\lambda - y = \mathbf{0}_n \\ Ax - b = \mathbf{0}_m \end{cases} \tag{10}$$

in which p is equal to m. If the triplet (x^*, y^*, λ^*) solves (10), then under some conditions, like for instance the convexity of the cost function c, the primal component x^* is a solution to the optimization problem (9).

1.4 Lorentz eigenvalue problem

The Lorentz eigenvalue problem is formulated in [26] in the following terms:

$$\begin{cases} \text{given a matrix } A \in \mathbb{M}_n, \\ \text{find } \lambda \in \mathbb{R} \text{ and a nonzero vector } x \in \mathbb{R}^n \text{ such that} \\ \mathbf{0}_n \preceq x \perp (Ax - \lambda x) \succeq \mathbf{0}_n. \end{cases} \tag{11}$$

By Corollary 2.1 in [24] one knows that (11) admits always a solution. If (x^*, λ^*) solves (11), then x^* is called a Lorentz eigenvector of A and λ^* is an associated Lorentz eigenvalue.

By a positive homogeneity argument, there is no loss of generality in assuming that x^* satisfies the equation $\langle e_n, x\rangle = 1$. Such normalization condition guarantees that x^* is a nonzero vector. So, the Lorentz eigenvalue problem (11) is nothing but the particular LEM

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ Ax - \lambda x - y = \mathbf{0}_n \\ \langle e_n, x\rangle - 1 = 0. \end{cases} \tag{12}$$

Theoretical aspects of the Lorentz eigenvalue problem has been discussed in [26]. Numerical methods for solving (12) has been proposed in [1] and [20, Section 2.2].

1.5 Lorentz quadratic eigenvalue problem

The general theory of cone-constrained quadratic eigenvalue problems is developed in [25]. The quadratic version of (11) involves a matrix

$$\mathcal{Q}(\lambda) := \lambda^2 A + \lambda B + C \tag{13}$$

that depends quadratically on the variable $\lambda \in \mathbb{R}$. One refers to the function $\mathcal{Q} : \mathbb{R} \to \mathbb{M}_n$ as the quadratic pencil associated to the triplet

$$(A, B, C) \in \mathbb{T}_n := \mathbb{M}_n \times \mathbb{M}_n \times \mathbb{M}_n.$$

The vector space of quadratic pencils is identified with $\mathbb{T}_n$. The formulation of the Lorentz quadratic eigenvalue problem reads as follows:

$$\begin{cases} \text{given a quadratic pencil } \mathcal{Q} \in \mathbb{T}_n, \\ \text{find } \lambda \in \mathbb{R} \text{ and a nonzero vector } x \in \mathbb{R}^n \text{ such that} \\ \mathbf{0}_n \preceq x \perp \mathcal{Q}(\lambda)x \succeq \mathbf{0}_n. \end{cases} \tag{14}$$

If (x^*, λ^*) is a solution to the above problem, then x^* is called a Lorentz eigenvector of $\mathcal{Q}$ and λ^* is an associated Lorentz eigenvalue. Of course, one is concerned here with the particular LEM

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ \mathcal{Q}(\lambda)x - y = \mathbf{0}_n \\ \langle e_n, x\rangle - 1 = 0. \end{cases} \tag{15}$$

The problems (11) and (14) look similar, but they differ in a number of aspects. For instance, the problem (11) admits always a solution, whereas (14) may not be solvable.

2 An existence result for LEMs of explicit type

A LEM is of explicit type if f has the special structure $f(x, y, \lambda) = h(x, \lambda) - y$, where $h : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is a smooth function. In such case the dual variables can be expressed explicitly in terms of the primal and exogenous variables. The next proposition is an existence result for the explicit LEM

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ h(x, \lambda) - y = \mathbf{0}_n \\ g(x) = \mathbf{0}_m. \end{cases} \tag{16}$$

Its proof is based on a celebrated theorem by Ky Fan on the existence of solutions to variational inequalities.

Proposition 2.1 *The combination of the following two hypotheses ensures that* (16) *is solvable:*

(i) $\Omega := \{x \in \mathbb{L}_n : g(x) = \mathbf{0}_m\}$ *is a convex compact set such that* $\mathbb{L}_n = \{tw : t \geq 0, w \in \Omega\}$.
(ii) For all $x \in \Omega$*, the equation* $\langle x, h(x, \lambda)\rangle = 0$ *has a solution* λ_x *that depends continuously on* x.

Proof Recall that h is assumed to be smooth. In particular, h is continuous. Consider the function $\Phi : \Omega \times \Omega \to \mathbb{R}$ given by

$$\Phi(x, w) = \langle w, h(x, \lambda_x)\rangle .$$

Since Ω is a convex compact set and

$$\begin{cases} \text{for all } w \in \Omega, \ \Phi(\cdot, w) \text{ is continuous,} \\ \text{for all } x \in \Omega, \ \Phi(x, \cdot) \text{ is linear,} \\ \text{for all } x \in \Omega, \ \Phi(x, x) = 0, \end{cases}$$

Ky Fan's theorem (cf. [4, Theorem 3.1.1]) ensures the existence of a point $x^* \in \Omega$ such that $\Phi(x^*, w) \geq 0$ for all $w \in \Omega$. If one sets $\lambda^* = \lambda_{x^*}$ and $y^* = h(x^*, \lambda^*)$, then it is not difficult to check that (x^*, y^*, λ^*) is a solution to (16). The details are omitted. □

The hypothesis (i) is strong in general. For instance, it does not hold for the explicit LEM (7) because the set

$$\{x \in \mathbb{L}_n : \|x\|^2 - 1 = 0\}$$

is not convex. However, the hypothesis (i) is in force for the explicit LEMs (12) and (15). The next corollary is obtained as a consequence of Proposition 2.1 or by specializing [25, Theorem 3.3] to the case of the Lorentz cone.

Corollary 2.2 *Let* $\mathcal{Q} \in \mathbb{T}_n$ *be a quadratic pencil as in* (13)*. Suppose that*

$$\langle x, Ax\rangle \neq 0 \quad \textit{for all } x \succeq \mathbf{0}_n, \ x \neq \mathbf{0}_n$$

and that $\mathcal{Q}$ *is Lorentz hyperbolic in the sense that*

$$\langle x, Bx\rangle^2 \geq 4\langle x, Ax\rangle\langle x, Cx\rangle \quad \textit{for all } x \succeq \mathbf{0}_n.$$

Then (14) *admits a solution.*

3 The companion system associated to a LEM

The main source of difficulties in a LEM is the presence of the nonnegativity constraints $x \succeq \mathbf{0}_n$ and $y \succeq \mathbf{0}_n$. Fortunately, it is possible to get rid of these constraints by introducing the change of variables

$$x \ = \ u \circ u \ = \left(2r\xi, \|\xi\|^2 + r^2\right), \tag{17}$$
$$y \ = \ v \circ v \ = \left(2s\eta, \|\eta\|^2 + s^2\right), \tag{18}$$

where $\circ$ is a vector product on $\mathbb{R}^n$ defined by

$$u \circ v \ := \ (r\eta + s\xi, \langle \xi, \eta\rangle + rs) .$$

In the above line and in the sequel one uses the decomposition

$$u = (\xi, r), \quad v = (\eta, s)$$

with $\xi, \eta \in \mathbb{R}^{n-1}$ and $r, s \in \mathbb{R}$. It is well known (cf. [8, Example 2.0]) that the triplet $(\mathbb{R}^n, \langle \cdot, \cdot \rangle, \circ)$ satisfies the axioms of an Euclidean Jordan algebra and that

$$\mathbb{L}_n = \{w^{[2]} : w \in \mathbb{R}^n\} \tag{19}$$

with $w^{[2]} := w \circ w$. Hence, LEM(f, g) admits the equivalent formulation

$$\begin{cases} \langle u^{[2]}, v^{[2]} \rangle &= 0 \\ f(u^{[2]}, v^{[2]}, \lambda) &= \mathbf{0}_n \\ g(u^{[2]}) &= \mathbf{0}_m. \end{cases} \tag{20}$$

A result from the theory of Euclidean Jordan algebras (cf. [8, Proposition 6]) asserts that the scalar equation $\langle u^{[2]}, v^{[2]} \rangle = 0$ can be written in the vector form $u^{[2]} \circ v^{[2]} = \mathbf{0}_n$. Hence, (20) can be reformulated as

$$\begin{cases} u^{[2]} \circ v^{[2]} &= \mathbf{0}_n \\ f(u^{[2]}, v^{[2]}, \lambda) &= \mathbf{0}_n \\ g(u^{[2]}) &= \mathbf{0}_m. \end{cases} \tag{21}$$

By the way, if $p = m$ (as occurs quite often in practice), then (21) is a square system, i.e., the number of equations is equal to the number of unknown variables.

Note that (21) fits into the abstract framework of a system of nonlinear equations

$$\Phi(z) = \mathbf{0}, \tag{22}$$

where $\mathbf{0}$ is a zero vector of appropriate size and $\Phi : \mathcal{Z} \to \mathcal{W}$ is a smooth function between Euclidean spaces of possible different dimensions. When it comes to solve (22) numerically, there are two cases for consideration:

- *Square case*. If $\dim(\mathcal{W}) = \dim(\mathcal{Z})$, then (22) is a square system, which can be solved by initializing Newton's method

$$z^{\tau+1} = z^\tau - [\Phi'(z^\tau)]^{-1} \Phi(z^\tau)$$

  at a suitable starting point z^0. Here $\Phi'(z^\tau) : \mathcal{Z} \to \mathcal{W}$ is the differential of Φ at z^τ.
- *Underdetermined case*. If $\dim(\mathcal{W}) < \dim(\mathcal{Z})$, then (22) is underdetermined, i.e., the number of unknown variables is greater than the number of equations. In such a situation one uses the Normal Flow Algorithm (NFA)

$$z^{\tau+1} = z^\tau - [\Phi'(z^\tau)]^\dagger \Phi(z^\tau),$$

  where $L^\dagger : \mathcal{W} \to \mathcal{Z}$ denotes the Moore-Penrose inverse of a linear map $L : \mathcal{Z} \to \mathcal{W}$. A detailed convergence analysis of the NFA can be found in [28,29].

The general theory of nonlinear equations tells us that an element z^* of the solution set

$$\Phi^{-1}(\mathbf{0}) = \{z \in \mathcal{Z} : \Phi(z) = \mathbf{0}\}$$

is difficult to detect numerically if the differential map $\Phi'(z^*)$ is nonsurjective. In particular, for the system (21) it will be difficult to detect a solution $z^* = (u^*, v^*, \lambda^*)$ with $v^* = \mathbf{0}_n$. Indeed, the corresponding differential map $\Phi'(u^*, \mathbf{0}_n, \lambda^*)$ is clearly nonsurjective.

The phenomenon of lack of surjectivity in (21) can be remediated by shifting the attention to a slightly modified system, namely

$$\begin{cases} u \circ v & = \mathbf{0}_n \\ f(u^{[2]}, v^{[2]}, \lambda) & = \mathbf{0}_n \\ g(u^{[2]}) & = \mathbf{0}_m. \end{cases} \tag{23}$$

One refers to (23) as the *companion system* associated to LEM(f, g). As we shall see in Lemma 3.3, the equality $\langle u^{\langle 2\rangle}, v^{\langle 2\rangle}\rangle = 0$ implies that $u \circ v = \mathbf{0}_n$. Since the reverse implication is false, a solution to (23) does not solve necessarily (20).

Definition 3.1 If (u, v, λ) solves (23), but $x = u^{[2]}$ and $y = v^{[2]}$ are not orthogonal, then one refers to the triplet (x, y, λ) as a fake solution to LEM(f, g). Fake solutions to LEM(f) are defined in a similar way.

Example 3.2 By way of illustration we display a fake solution for the LCP stated in (6) with

$$M = \begin{bmatrix} 1 & 4 & \sqrt{5} \\ 2 & 5 & -1 \\ 3 & 6 & 3 \end{bmatrix}, \quad q = \begin{bmatrix} -5 \\ \sqrt{5} \\ -\sqrt{5} \end{bmatrix}.$$

Here $n = 3$ and $p = 0$. The pair $(u, v) = ((-1, 2, 0), (4, 2, 0))$ solves the companion system

$$\begin{cases} u \circ v & = \mathbf{0}_3 \\ Mu^{[2]} + q - v^{[2]} & = \mathbf{0}_3, \end{cases}$$

but $u^{[2]} = \left(0, 0, \sqrt{5}\right)$ and $v^{[2]} = \left(0, 0, 2\sqrt{5}\right)$ are not orthogonal.

The above example has been constructed artificially. A vast majority of solutions to the companion system (23) satisfy the orthogonality condition $\langle u^{[2]}, v^{[2]}\rangle = 0$. Said in other words, getting a fake solution to a LEM is an exception and not the general rule. This theme is developed next.

3.1 On fake solutions

We start by stating a technical lemma.

Lemma 3.3 *For all $u, v \in \mathbb{R}^n$, the orthogonality condition $\langle u^{[2]}, v^{[2]}\rangle = 0$ implies $u \circ v = \mathbf{0}_n$. Furthermore, the following statements are equivalent:*

(a) $u \circ v = \mathbf{0}_n$, but $u^{[2]}$ and $v^{[2]}$ are not orthogonal.
(b) $\xi, \eta \in \mathbb{R}^{n-1}$ are nonzero orthogonal vectors and $r = s = 0$.

Proof The equality $\langle u^{[2]}, v^{[2]}\rangle = 0$ says that

$$4rs\langle \xi, \eta\rangle + \left(\|\xi\|^2 + r^2\right)\left(\|\eta\|^2 + s^2\right) = 0.$$

This can be rearranged as

$$\|\xi\|^2\|\eta\|^2 - \langle \xi, \eta\rangle^2 + \|r\eta + s\xi\|^2 + (\langle \xi, \eta\rangle + rs)^2 = 0.$$

Since $\langle \xi, \eta\rangle \leq \|\xi\|\,\|\eta\|$, the above equality breaks down into three pieces:

$$\|\xi\|^2\|\eta\|^2 - \langle \xi, \eta\rangle^2 = 0 \tag{24}$$

$$r\eta + s\xi = 0 \tag{25}$$

$$\langle \xi, \eta\rangle + rs = 0. \tag{26}$$

Note that (25)–(26) says that $u \circ v = \mathbf{0}_n$. It is possible to have (25)–(26) without having (24). This happens exactly when ξ, η are nonzero orthogonal vectors and $r = s = 0$. □

Fake solutions are easily recognizable. In fact, one has:

Proposition 3.4 *If (x, y, λ) is a fake solution to a* LEM*, then x and y are nonzero vectors on the central axis of $\mathbb{L}_n$.*

Proof Let (x, y, λ) be a fake solution to a LEM. By Lemma 3.3 one has $(x, y) = \left(u^{[2]}, v^{[2]}\right)$ with $u = (\xi, 0)$, $v = (\eta, 0)$, and $\xi, \eta \in \mathbb{R}^{n-1}\backslash\{\mathbf{0}_{n-1}\}$ mutually orthogonal. Hence, $x = \left(\mathbf{0}_{n-1}, \|\xi\|^2\right)$ and $y = \left(\mathbf{0}_{n-1}, \|\eta\|^2\right)$ are nonzero vectors on the central axis of $\mathbb{L}_n$. □

By way of illustration, consider the Lorentz eigenvalue problem as formulated in (12). The system (21) becomes

$$\begin{cases} u^{[2]} \circ v^{[2]} &= \mathbf{0}_n \\ (A - \lambda I_n)u^{[2]} - v^{[2]} &= \mathbf{0}_n \\ \langle e_n, u^{[2]}\rangle - 1 &= 0, \end{cases} \tag{27}$$

where I_n is the identity matrix of order n. This is a system of $2n + 1$ equations in the same number of unknown variables. The companion counterpart

$$\begin{cases} u \circ v &= \mathbf{0}_n \\ (A - \lambda I_n)u^{[2]} - v^{[2]} &= \mathbf{0}_n \\ \langle e_n, u^{[2]}\rangle - 1 &= 0 \end{cases} \tag{28}$$

is a system of the same size. Unless the matrix $A \in \mathbb{M}_n$ is constructed in an artificial way, the systems (27) and (28) have the same solution set. A probabilistic formulation of this statement reads as follows:

Proposition 3.5 *Let $A \in \mathbb{M}_n$ be a random matrix with absolutely continuous probability distribution. Then, almost surely, the Lorentz eigenvalue problem (12) is free of fake solutions.* □

Proof If (x, y, λ) is a fake solution to (12), then $x = e_n$ and $y = \alpha\, e_n$ with $\alpha > 0$. The second equation in (12) becomes $Ae_n = (\alpha + \lambda)e_n$. Hence, the last column of A is a multiple of e_n, i.e.,

$$a_{1,n} = 0,\ a_{2,n} = 0, \ldots, a_{n-1,n} = 0.$$

Such event occurs with probability zero because $A \in \mathbb{M}_n$ is a random matrix with absolutely continuous probability distribution. □

3.2 Numerical tests with the companion system

The programming language used for all numerical tests was Scilab. The experiment reported in Table 1 compares Newton's method applied to (27) and Newton's method applied to the companion counterpart (28). The idea is measuring the rate of success as function of n and number of initial points. A few practical considerations are in order:

– An iterative algorithm like Newton's method (or the NFA) keeps running until one of the following three termination criteria occurs:

$$\begin{array}{ll} \tau = 1000 & \text{(maximum number of iterations)}, \\ \kappa(\Phi'(z^\tau)) \geq 10^5 & \text{(ill-conditioning)}, \\ \|\Phi(z^\tau)\| \leq 10^{-8} & \text{(a solution is found)}. \end{array}$$

Table 1 Solving a Lorentz eigenvalue problem by using Newton's method

n	Newton's method applied to (27)			Newton's method applied to (28)		
	NIP = 1 (%)	NIP = 10 (%)	NIP = 10^2 (%)	NIP = 1 (%)	NIP = 10 (%)	NIP = 10^2 (%)
5	7.4	58.0	99.9	45.7	100	100
10	0.1	5.1	33.5	20.0	88.1	100
15	0	0.1	1.0	8.9	63.2	100
20	0	0	0	6.0	40.6	99.6

Percentages of success are estimated with a sample of 10^3 random matrices A with Gaussian distribution

Here $\kappa(M) = \sigma_{\max}(M)/\sigma_{\min}(M)$ refers to the condition number of a matrix M.

- An iterative algorithm is declared successful if a solution is found while working with a certain tolerated Number of Initial Points (for instance, NIP $= 1$, NIP $= 10$, or NIP $= 10^2$).
- In general, there is no optimal rule for selecting a suitable initial point. For the systems (27) and (28), the initial point $z^0 = (u^0, v^0, \lambda^0)$ is constructed as follows:

$$\begin{cases} u^0 = \text{ uniformly distributed on the unit sphere of } \mathbb{R}^n \\ x^0 = (u^0)^{[2]} \\ \lambda^0 = \langle x^0, Ax^0 \rangle / \|x^0\|^2 \\ v^0 = n\text{-dimensional Gaussian vector} \\ z^0 = (u^0, v^0, \lambda^0). \end{cases}$$

A Gaussian vector (or matrix) is a vector (or matrix) whose the entries are independent random variables with standard normal distributions.

As one can see from Table 1, Newton's method performs poorly on (27). To understand why the percentages of success are so low, recall that Newton's method is unsuccessful if one gets a current point z^τ at which $\Phi'(z^\tau)$ is ill-conditioned. This is exactly what is happening while handling the ill-conditioned system (27). Newton's method performs much better on the companion system (28).

Remark 3.6 The performance of Newton's method on (27) does not improve if one considers higher ill-conditioning conditions. We did a few numerical experiments with $\kappa(\Phi'(z^\tau)) \geq 10^8$ and even with $\kappa(\Phi'(z^\tau)) \geq 10^{16}$, but the percentages of success remain essentially unchanged.

The numerical experiment reported in Table 3 is different in spirit. The purpose of this experiment is to see whether Newton's method on (28) tends to favor a certain type of solution, be it central, eccentric, or boundary type. One considers the test matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 6 & 0 & 0 \\ -2 & 0 & 5 & 0 \\ -6 & 4 & 1 & 4 \end{bmatrix}, \tag{29}$$

which has exactly six Lorentz eigenvalues: one coming from a central solution, one coming from an eccentric solution, and four coming from boundary type solutions. The details are

Table 2 Lorentz eigenvalues of the matrix (29)

Type	λ	Primal vector				Dual vector			
		x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
Boundary	2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	1
Boundary	3	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	2
Central	4	0	0	0	1	0	0	0	0
Boundary	5	0	0	1	1	0	0	0	0
Eccentric	6	0	$\frac{1}{2}$	0	1	0	0	0	0
Boundary	7	0	1	0	1	0	-1	0	1

Table 3 Finding the Lorentz eigenvalues of the matrix (29) by applying Newton's method to the companion system (28)

Ill-conditioning	Divergence	Central	Eccentric	Boundary			
				$\lambda = 2$	$\lambda = 3$	$\lambda = 5$	$\lambda = 7$
55.0 %	0.0 %	3.6 %	2.7 %	21.6 %	15.4 %	0 %	1.7 %

The percentages of failure (ill-conditioning or divergence) and the percentages of converge to the different solutions are estimated with a random sample of 10^4 initial points

displayed in Table 2. Beware that the classical eigenvalue $\lambda = 1$ is not a Lorentz eigenvalue because the associated eigenvector $x = (2, 2, 1, 1)$ is not in the Lorentz cone. Conversely, the Lorentz eigenvalues $\lambda = 2, \lambda = 3$, and $\lambda = 7$ are not eigenvalues in the classical sense because the corresponding dual vectors y are nonzero. Table 3 shows that the Lorentz eigenvalue $\lambda = 2$ is the most likely to be detected, then comes the Lorentz eigenvalue $\lambda = 3$, and so on. In this numerical experiment the particular solution

$$z^* = (x^*, y^*, \lambda^*) = ((0, 0, 1, 1), (0, 0, 0, 0), 5)$$

was never detected. This fact can be explained by recalling that Newton's method is not suitable for finding $z^* \in \Phi^{-1}(0)$ such that $\Phi'(z^*)$ is nonsurjective.

4 The use of natural coordinates

We come back again to the squaring technique based on the change of variables (17)–(18) and characterize the solutions to a LEM in terms of the "natural" coordinates ξ, r, η, s, λ. For the sake of clarity in the presentation, we distinguish between trivial and nontrivial solutions.

4.1 The trivial case

Searching for a trivial solution to a LEM makes sense only if $g(\mathbf{0}_n) = \mathbf{0}_m$ or if the function g is missing altogether. Since the component x of a trivial solution is required to be zero, one must substitute $u = \mathbf{0}_n$ into (20). Such a substitution leads to

$$f_\sharp(\eta, s, \lambda) = \mathbf{0}_n, \tag{30}$$

where $f_\sharp : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is the smooth function given by

$$f_\sharp(\eta, s, \lambda) = f\left(\mathbf{0}_n, \left(2s\eta, \|\eta\|^2 + s^2\right), \lambda\right).$$

Note that (30) is a square system if $p = 0$ and it is an underdetermined system if $p \geq 1$.

Example 4.1 By way of illustration, consider the feasibility problem:

$$\begin{cases} \text{given smooth functions } M : \mathbb{R} \to \mathbb{M}_n \text{ and } q : \mathbb{R} \to \mathbb{R}^n, \\ \text{find } \lambda \in \mathbb{R} \text{ and } y \succeq \mathbf{0}_n \text{ such that } M(\lambda)\, y = q(\lambda). \end{cases} \tag{31}$$

In other words, one needs to find a value of the parameter λ so that the parametric linear system $M(\lambda)\, y = q(\lambda)$ has a solution y in the Lorentz cone. One may see (31) as the problem of finding a trivial solution to the particular LEM

$$\begin{cases} \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n \\ M(\lambda)\, y - q(\lambda) = \mathbf{0}_n. \end{cases}$$

There is no normalization condition on the primal variables. Finding a solution to the feasibility problem (31) is then a matter of solving (30) with

$$f_\sharp(\eta, s, \lambda) = M(\lambda) \begin{bmatrix} 2s\eta \\ \|\eta\|^2 + s^2 \end{bmatrix} - q(\lambda).$$

What we are doing, in essence, is to express y in terms of the coordinates η and s. In this example one has $p = 1$, and therefore one must solve an underdetermined system of nonlinear equations.

4.2 The nontrivial case

This case is the most interesting one, but it must be handled with care. For the sake of simplicity, from now on we assume that

$$g(\mathbf{0}_n) \neq \mathbf{0}_m. \tag{32}$$

Such hypothesis ensures that every solution to LEM(f, g) is nontrivial. Note that (32) is in force for the particular LEM's (7), (12), and (15).

Lemma 4.2 *Consider the change of variables* (17)–(18). *Under the assumption* (32)*, the triplet* (x, y, λ) *is a solution to* LEM(f, g) *if and only if one of the following situations occur:*

(a) $r \neq 0$ *and* $(\xi, r, \eta, s, \lambda)$ *solves the companion system*

$$\begin{cases} r\eta + s\xi & = \mathbf{0}_{n-1} \\ \langle \xi, \eta \rangle + rs & = 0 \\ F(\xi, r, \eta, s, \lambda) & = \mathbf{0}_n \\ G(\xi, r) & = \mathbf{0}_m, \end{cases} \tag{33}$$

where

$$F(\xi, r, \eta, s, \lambda) = f\left(\left(2r\xi, \|\xi\|^2 + r^2\right), \left(2s\eta, \|\eta\|^2 + s^2\right), \lambda\right)$$
$$G(\xi, r) = g\left(2r\xi, \|\xi\|^2 + r^2\right).$$

(b) $r = 0$, $\eta = \mathbf{0}_{n-1}$, $s = 0$, *and* (ξ, λ) *satisfies*

$$\begin{cases} f\left(\left(\mathbf{0}_{n-1}, \|\xi\|^2\right), \mathbf{0}_n, \lambda\right) = \mathbf{0}_n \\ g\left(\mathbf{0}_{n-1}, \|\xi\|^2\right) \qquad\qquad = \mathbf{0}_m. \end{cases} \tag{34}$$

Proof As we saw already in the proof of Lemma 3.3, the orthogonality condition in (20) is equivalent to (24)–(26). Hence, (x, y, λ) solves (4) if and only if $(\xi, r, \eta, s, \lambda)$ satisfies (24)–(26) and

$$\begin{cases} F(\xi, r, \eta, s, \lambda) = \mathbf{0}_n \\ G(\xi, r) \qquad\quad = \mathbf{0}_m. \end{cases}$$

Depending on the value of r, there are two cases for consideration. If $r \neq 0$, then the Eq. (24) becomes redundant because it can be deduced from (25). One ends up with the system (33). If $r = 0$, then the assumption (32) implies that $\xi \neq \mathbf{0}_{n-1}$. Hence, (24)–(26) is equivalent to saying that $\eta = \mathbf{0}_{n-1}$ and $s = 0$. One ends up with the system (34). □

The new system (33) is the same, of course, as the old companion system (23), except that now everything is expressed in terms of the natural coordinates.

4.3 Boundary type and interior type solutions

The next theorem is the main result of Sect. 4.

Theorem 4.3 *Let the assumption* (32) *be in force. The triplet* (x, y, λ) *solves* LEM(f, g) *if and only if one of the following situations occur:*

(α) $x = \left(\mathbf{0}_{n-1}, \|\xi\|^2\right)$, $y = \mathbf{0}_n$, *and* (ξ, λ) *solves* (34).

(β) $x = 2r^2(\omega, 1)$, $y = 2s^2(-\omega, 1)$, *and* (ω, r, s, λ) *solves*

$$\begin{cases} \mathcal{F}_{\text{bd}}(\omega, r, s, \lambda) = \mathbf{0}_n \\ \mathcal{G}_{\text{bd}}(\omega, r) \qquad = \mathbf{0}_m \\ \|\omega\|^2 - 1 \qquad = 0 \end{cases} \tag{35}$$

where

$$\mathcal{F}_{\text{bd}}(\omega, r, s, \lambda) = f\left(2r^2(\omega, 1), 2s^2(-\omega, 1), \lambda\right)$$
$$\mathcal{G}_{\text{bd}}(\omega, r) = g\left(2r^2(\omega, 1)\right).$$

(γ) $x = r^2\left(2\omega, 1 + \|\omega\|^2\right)$, $y = \mathbf{0}_n$, *and* (ω, r, λ) *solves*

$$\begin{cases} \mathcal{F}(\omega, r, \lambda) = \mathbf{0}_n \\ \mathcal{G}(\omega, r) \qquad = \mathbf{0}_m, \end{cases} \tag{36}$$

where

$$\mathcal{F}(\omega, r, \lambda) = f\left(r^2\left(2\omega, 1 + \|\omega\|^2\right), \mathbf{0}_n, \lambda\right)$$
$$\mathcal{G}(\omega, r) = g\left(r^2\left(2\omega, 1 + \|\omega\|^2\right)\right).$$

Proof Consider again the change of variables (17)–(18). Let us examine more closely the part (a) of Lemma 4.2. The condition $r \neq 0$ and the first equation in (33) yield $\eta = -s\omega$ with $\omega = \xi/r$. Hence, Lemma 4.2(a) says that

$$x = r^2\left(2\omega, 1 + \|\omega\|^2\right), \tag{37}$$
$$y = s^2\left(-2\omega, 1 + \|\omega\|^2\right), \tag{38}$$

where (ω, r, s, λ) solves the following reduced form of the companion system:

$$s\left(1-\|\omega\|^2\right)=0 \tag{39}$$

$$f\left(r^2\left(2\omega, 1+\|\omega\|^2\right), s^2\left(-2\omega, 1+\|\omega\|^2\right), \lambda\right)=\mathbf{0}_n \tag{40}$$

$$g\left(r^2\left(2\omega, 1+\|\omega\|^2\right)\right)=\mathbf{0}_m. \tag{41}$$

Note that we got rid of η and changed ξ by ω. The Eq. (39) says that either $\|\omega\|=1$ or $s=0$. By plugging $\|\omega\|=1$ into (37)–(38) and (40)–(41) one gets (β). By plugging $s=0$ into (38) and (40)–(41) one gets (γ). □

The case (α) is disjoint from (β) and also disjoint from (γ). However, the cases (β) and (γ) are not disjoint. Indeed, by setting $s=0$ in (β) one recovers the same solutions as those obtained by setting $\|\omega\|=1$ in (γ). We end this section by giving a full classification of the solutions to LEM(f, g). The main merit of the next corollary is to provide a clear-cut characterization of the boundary type solutions.

Corollary 4.4 *Let the assumption* (32) *be in force. Then*

(a) (x, y, λ) *is a central solution to* LEM(f, g) *if and only if* (α) *occurs.*
(b) (x, y, λ) *is a boundary type solution to* LEM(f, g) *if and only if* (β) *occurs.*
(c) (x, y, λ) *is an eccentric solution to* LEM(f, g) *if and only if* (γ) *occurs with* $\|\omega\| \notin \{0, 1\}$.
□

Proof Part (a). If (α) occurs, then (x, y, λ) is a central solution because $x=\left(\mathbf{0}_{n-1}, \|\xi\|^2\right)$ lies on the central axis of the Lorentz cone. Note that $\xi \neq \mathbf{0}_{n-1}$ thanks to (32). Conversely, if (x, y, λ) is a central solution, then y is necessarily equal to $\mathbf{0}_n$. Furthermore, $x=(\mathbf{0}_{n-1}, t)$ with t being a positive scalar such that

$$\begin{cases} f\left((\mathbf{0}_{n-1}, t), \mathbf{0}_n, \lambda\right)=\mathbf{0}_n \\ g\left(\mathbf{0}_{n-1}, t\right) \qquad\qquad =\mathbf{0}_m. \end{cases}$$

It suffices then to write $t=\|\xi\|^2$ for a suitable $\xi \in \mathbb{R}^{n-1}$.

Part (b). If (β) occurs, then (x, y, λ) is a boundary type solution. Indeed,

$$x_n-\left[x_1^2+\cdots+x_{n-1}^2\right]^{1/2} \;=\; 2r^2(1-\|\omega\|) \;=\; 0$$

thanks to the last equation in (35). Conversely, suppose that (x, y, λ) is a boundary type solution. Let x and y be as in (17)–(18). The case $r=0$ must be excluded, so one can rewrite x and y as in (37)–(38). Since $x \in \mathrm{bd}(\mathbb{L}_n)$ and $r \neq 0$, one deduces that

$$\left(2\omega, 1+\|\omega\|^2\right) \in \mathrm{bd}(\mathbb{L}_n).$$

But this amounts to saying that $\|2\omega\|=1+\|\omega\|^2$, which in turn is equivalent to $\|\omega\|=1$. One arrives in this way to the condition (β).

Part (c). If (γ) occurs with $\|\omega\| \notin \{0, 1\}$, then (x, y, λ) is an eccentric solution. Indeed

$$\left(2\omega, 1+\|\omega\|^2\right) \notin \mathrm{bd}(\mathbb{L}_n),$$

and $x=r^2\left(2\omega, 1+\|\omega\|^2\right)$ belongs to the interior of $\mathbb{L}_n$. Furthermore, x is not on the central axis of $\mathbb{L}_n$ because $\omega \neq \mathbf{0}_{n-1}$. Conversely, let (x, y, λ) be an eccentric solution. The cases (α) and (β) must be excluded, and therefore (γ) occurs. Since x is in the interior of the Lorentz cone, but not on its central axis, one necessarily has $\|\omega\| \notin \{0, 1\}$. □

5 Analysis of the Lorentz quadratic eigenvalue problem

Testing Lorentz hyperbolicity is not a trivial matter. Recall, from Corollary 2.2, that a quadratic pencil $\mathcal{Q}$ is Lorentz hyperbolic if the discriminant function

$$x \in \mathbb{R}^n \mapsto \delta_{\mathcal{Q}}(x) = \langle x, Bx\rangle^2 - 4\langle x, Ax\rangle\langle x, Cx\rangle$$

is nonnegative on $\mathbb{L}_n$. Note that δ_Q is an homogeneous multivariate polynomial of degree 4. Hence,

$$u \in \mathbb{R}^n \mapsto \Psi_{\mathcal{Q}}(u) = \delta_{\mathcal{Q}}(u^{[2]})$$

is an homogeneous multivariate polynomial of degree 8. The representation (19) of the cone $\mathbb{L}_n$ leads straightforwardly to the following characterization of Lorentz hyperbolicity.

Proposition 5.1 *A quadratic pencil $\mathcal{Q}$ is Lorentz hyperbolic if and only if the homogeneous polynomial $\Psi_{\mathcal{Q}}$ is nonnegative.*

The polynomial $\Psi_{\mathcal{Q}}$ is certainly nonnegative if it can be written as a sum of squares (SOS) of other polynomials in u. Hence, the existence of a SOS representation for $\Psi_{\mathcal{Q}}$ is a sufficient condition for $\mathcal{Q}$ to be Lorentz hyperbolic. According to the specialized literature [7,12,19], checking whether a given homogeneous multivariate polynomial admits a SOS representation reduces to solving a semidefinite program, a convex optimization problem that one may solve efficiently to arbitrary precision, in polynomial time in the input size.

5.1 Boundary type solutions to the Lorentz quadratic eigenvalue problem

For the particular LEM given in (15) one has

$$\mathcal{F}_{\text{bd}}(\omega, r, s, \lambda) = 2r^2\mathcal{Q}(\lambda)\begin{bmatrix}\omega\\1\end{bmatrix} - 2s^2\begin{bmatrix}-\omega\\1\end{bmatrix}$$
$$\mathcal{G}_{\text{bd}}(\omega, r) = 2r^2 - 1.$$

The boundary type solutions to (15) are therefore found by solving

$$\begin{cases} 2r^2\mathcal{Q}(\lambda)\begin{bmatrix}\omega\\1\end{bmatrix} - 2s^2\begin{bmatrix}-\omega\\1\end{bmatrix} = \mathbf{0}_n \\ 2r^2 - 1 = 0 \\ \|\omega\|^2 - 1 = 0. \end{cases}$$

By getting rid of r and by making the change of variables $\mu = \sqrt{2}s$, one obtains

$$\begin{cases} \mathcal{Q}(\lambda)\begin{bmatrix}\omega\\1\end{bmatrix} - \mu^2\begin{bmatrix}-\omega\\1\end{bmatrix} = \mathbf{0}_n \\ \|\omega\|^2 - 1 = 0. \end{cases} \tag{42}$$

Remark 5.2 If one introduces the diagonal matrix

$$J_n := \text{Diag}(1, \ldots, 1, -1) = \begin{bmatrix} I_{n-1} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & -1 \end{bmatrix},$$

Table 4 Finding boundary type solutions to the Lorentz quadratic eigenvalue problem

n	Newton's method applied to (42)		
	NIP = 1 (%)	NIP = 10 (%)	NIP = 10^2 (%)
5	63.0	97.4	98.1
10	32.4	83.4	96.7
15	25.3	72.5	96.1
20	19.5	65.7	95.8

Percentages of success are estimated with a sample of 10^3 random quadratic pencils $\mathcal{Q} \in \mathbb{T}_n$

then one can reformulate (42) as a classical quadratic eigenvalue problem

$$\begin{cases} \mathcal{Q}_\mu(\lambda) \begin{bmatrix} \omega \\ 1 \end{bmatrix} = \mathbf{0}_n \\ \|\omega\|^2 - 1 \quad = 0 \end{cases} \tag{43}$$

for the pencil $\mathcal{Q}_\mu(\lambda) = \lambda^2 A + \lambda B + C + \mu^2 J_n$. There is a very rich literature devoted to the numerical analysis of classical quadratic eigenvalue problems, see for instance the books [11,16] or the survey paper [27]. A bothersome aspect of (43) is that the quadratic pencil $\mathcal{Q}_\mu$ depends on the unknown parameter μ.

The system (42) is smooth and concerns $n+1$ nonlinear equations in the same number of unknown variables. Table 4 illustrates the performance of Newton's method on (42). Some technical comments are in order. If (ω, μ, λ) solves (42), then clearly

$$\mu = \pm \left[\frac{1}{2} \left\langle \mathcal{Q}(\lambda) \begin{bmatrix} \omega \\ 1 \end{bmatrix}, \begin{bmatrix} -\omega \\ 1 \end{bmatrix} \right\rangle \right]^{1/2}.$$

Hence, a natural way to construct an initial point $z^0 = (\omega^0, \mu^0, \lambda^0)$ is as follows:

– one generates a random vector ω^0 with uniform distribution on the unit sphere of $\mathbb{R}^{n-1}$, a discrete random variable δ with uniform distribution on $\{-1, 1\}$, and a random variable λ^0 with uniform distribution on some interval $[a_n, b_n]$. Then one sets

$$\mu^0 = \delta \left[\frac{1}{2} \left\langle \mathcal{Q}\left(\lambda^0\right) \begin{bmatrix} \omega^0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\omega^0 \\ 1 \end{bmatrix} \right\rangle \right]^{1/2}.$$

For simplicity we take $a_n = -10$ and $b_n = 10$ for all n, but one may consider more sophisticated options.

Percentages of success are estimated by averaging the results obtained with a sample of 10^3 random quadratic pencils $\mathcal{Q} \in \mathbb{T}_n$. In order to enhance the chances of (42) to have a solution, each random quadratic pencil is constructed as follows: one generates Gaussian symmetric matrices $\tilde{A}, \tilde{B}, \tilde{C}$ and takes $\mathcal{Q}$ as the pencil associated to

$$(A, B, C) = \left(\Pi_{\mathrm{PSD}}(\tilde{A}), \tilde{B}, -\Pi_{\mathrm{PSD}}(-\tilde{C})\right),$$

where $\Pi_{\mathrm{PSD}}(\cdot)$ stands for the projection on the Loëwner cone of positive semidefinite symmetric matrices. Such $\mathcal{Q}$ is obviously Lorentz hyperbolic.

5.2 Interior type solutions to the Lorentz quadratic eigenvalue problem

Consider now the problem of finding interior type solutions to (15), be them central or eccentric. Both cases can be treated simultaneously. The system (36) takes here the form

$$\begin{cases} r^2 \mathcal{Q}(\lambda) \begin{bmatrix} 2\omega \\ 1 + \|\omega\|^2 \end{bmatrix} = \mathbf{0}_n \\ r^2 \left(1 + \|\omega\|^2\right) - 1 \quad = 0. \end{cases}$$

Since r cannot be equal to 0, the above system can we rewritten as

$$\begin{cases} \mathcal{Q}(\lambda) \begin{bmatrix} 2r^2\omega \\ 1 \end{bmatrix} \qquad = \mathbf{0}_n \\ r^2 \left(1 + \|\omega\|^2\right) - 1 = 0. \end{cases}$$

Returning to the original variable $\xi = r\omega$, one ends up with the equivalent system

$$\begin{cases} \mathcal{Q}(\lambda) \begin{bmatrix} 2r\xi \\ 1 \end{bmatrix} \quad = \mathbf{0}_n \\ \|\xi\|^2 + r^2 - 1 = 0. \end{cases} \tag{44}$$

The system (44) produces all the central solutions (obtained with $r\xi = 0$) and all the eccentric solutions (obtained with $r\xi \neq 0$, $\|\xi\|^2 \neq r^2$).

Remark 5.3 Beware that (44) may produce also a few boundary type solutions. To see this, just consider the configuration $r\xi \neq 0$, $\|\xi\|^2 = r^2$. Of course, when it comes to search for the whole collection of boundary type solutions it is better to shift the attention to the system (42).

6 Alternative techniques and extensions

6.1 Techniques of nonsmooth analysis

Another way of getting rid of the nonnegativity constraints $x \succeq \mathbf{0}_n$ and $y \succeq \mathbf{0}_n$ is to use a complementarity function associated to the Lorentz cone. This terminology refers to any function $\mathcal{C} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathcal{C}(x, y) = \mathbf{0}_n \quad \Leftrightarrow \quad \mathbf{0}_n \preceq x \perp y \succeq \mathbf{0}_n. \tag{45}$$

With the help of such a function $\mathcal{C}$ one can reformulate LEM(f, g) as a system

$$\begin{cases} \mathcal{C}(x, y) \quad\;\; = \mathbf{0}_n \\ f(x, y, \lambda) = \mathbf{0}_n \\ g(x) \qquad\;\; = \mathbf{0}_m \end{cases} \tag{46}$$

of $2n + m$ equations and $2n + p$ unknown variables.

The most popular choice of $\mathcal{C}$ is the Fischer–Burmeister complementarity function

$$\mathcal{C}_{\mathrm{FB}}(x, y) = x + y - \left[x^{[2]} + x^{[2]}\right]^{[1/2]},$$

where the square root operation $[\,\cdot\,]^{[1/2]}$ is understood in the Jordan algebra sense. Another choice of interest is the natural complementarity function

$$\mathcal{C}_{\mathrm{nat}}(x, y) = x - (x - y)_+.$$

A common feature of $\mathcal{C}_{\rm FB}$ and $\mathcal{C}_{\rm nat}$ is that both functions are locally Lipschitz and semismooth; see [5] for definitions and proofs. So, with any of the above choices of $\mathcal{C}$, the system (46) has the general structure (22) with $\Phi : \mathcal{Z} \to \mathcal{W}$ standing for a locally Lipschitz semismooth function between Euclidean spaces of possible different dimensions. The Newton method and the NFA take now a more general form, to wit:

- *Square case*. If $\dim(\mathcal{W}) = \dim(\mathcal{Z})$, then (22) is a square system. One solves it with the Semismooth Newton Method (SNM)

$$z^{\tau+1} = z^{\tau} - M_{\tau}^{-1}\Phi(z^{\tau}),$$

  where $M_{\tau} : \mathcal{Z} \to \mathcal{W}$ is a linear map taken from the Clarke generalized differential $\partial\Phi(z^{\tau})$ of Φ at z^{τ} (cf. [22]).
- *Underdetermined case*. If $\dim(\mathcal{W}) < \dim(\mathcal{Z})$, then (22) is underdetermined. In such a situation one uses the Semismooth Normal Flow Algorithm (SNFA)

$$z^{\tau+1} = z^{\tau} - M_{\tau}^{\dagger}\Phi(z^{\tau}),$$

  where M_{τ} is chosen arbitrarily from $\partial\Phi(z^{\tau})$.

By way of example, consider again the Lorentz eigenvalue problem as formulated in (12). The system (22) becomes

$$\begin{cases} \mathcal{C}(x, y) & = \mathbf{0}_n \\ (A - \lambda I_n)x - y & = \mathbf{0}_n \\ \langle e_n, x\rangle - 1 & = 0. \end{cases}$$

This is system of $2n + 1$ equations in the same number of unknown variables, so one can solve it by using the SNM. This idea has been suggested in [1, Section 4.1].

6.2 Equilibrium models constrained by elliptic cones

The Lorentz equilibrium model (4) can be seen as a particular instance of an equilibrium model

$$\begin{cases} K \ni x \perp y \in K^{+} \\ f(x, y, \lambda) = \mathbf{0}_n \\ g(x) = \mathbf{0}_m \end{cases} \tag{47}$$

in which the complementarity conditions are expressed in terms of a closed convex cone K. The notation K^{+} refers to the dual cone of K. The squaring technique developed in this work can be extended to the case in which

$$K = \mathcal{E}(Q) := \{(\omega, t) \in \mathbb{R}^n : \sqrt{\langle \omega, Q\omega\rangle} \le t\} \tag{48}$$

is an elliptic cone associated to a positive definite matrix Q of order $n-1$. Elliptic cones have been studied under different angles in a number of references. Complementarity conditions relative to an elliptic cone are considered in [3].

An elliptic cone can be seen as the image of the Lorentz cone under an invertible linear transformation. Indeed, one has

$$\mathcal{E}(Q) = \{S_Q x : x \succeq \mathbf{0}_n\} = \{S_Q u^{[2]} : u \in \mathbb{R}^n\},$$

where $S_Q : \mathbb{R}^n \to \mathbb{R}^n$ is the invertible linear transformation whose matrix representation is

$$S_Q = \begin{bmatrix} Q^{-1/2} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & 1 \end{bmatrix}.$$

One can easily check that the dual of (48) is given by

$$(\mathcal{E}(Q))^+ = \mathcal{E}(Q^{-1}) = \{S_{Q^{-1}} y : y \succeq \mathbf{0}_n\} = \{S_{Q^{-1}} v^{[2]} : v \in \mathbb{R}^n\}.$$

Hence, the equilibrium model (47) can be converted into a system of nonlinear equations, namely

$$\begin{cases} \langle S_Q u^{[2]}, S_{Q^{-1}} v^{[2]} \rangle & = 0 \\ f(S_Q u^{[2]}, S_{Q^{-1}} v^{[2]}, \lambda) & = \mathbf{0}_n \\ g(S_Q u^{[2]}) & = \mathbf{0}_m. \end{cases}$$

Since S_Q is symmetric and $S_{Q^{-1}} = (S_Q)^{-1}$, it is clear that

$$\left\langle S_Q u^{[2]}, S_{Q^{-1}} v^{[2]} \right\rangle = \left\langle u^{[2]}, v^{[2]} \right\rangle.$$

In other words, the seemingly more general model (47) is nothing but LEM(f_Q, g_Q) with

$$f_Q(x, y, \lambda) = f(S_Q x, S_{Q^{-1}} y, \lambda)$$
$$g_Q(x, y, \lambda) = g(S_Q x).$$

In short, the whole theory of Lorentz equilibrium models extends to the elliptic case.

6.3 Final comments

The leading motivation of this work has been to exploit the possibility of getting rid of the nonnegativity constraint $x \succeq \mathbf{0}_n$ by expressing the vector $x \in \mathbb{R}^n$ as the "square" of another vector $u \in \mathbb{R}^n$. Of course, the squaring technique is applied also to the nonnegativity constraint $y \succeq \mathbf{0}_n$ imposed on the dual variables. As an alternative to the squaring technique one may consider the use of a complementarity function as in (45). Some preliminary numerical experiments with the Lorentz eigenvalue problem suggest that the combination of the complementarity function technique and the SNM performs slightly better than the combination of the squaring technique and the classical Newton method. Having said this, we would like to point out that the squaring technique has at least two advantages:

- First of all, the squaring technique is well suited to discriminate between different types of solutions to a LEM (boundary type, interior type, et cetera).
- Secondly, the squaring technique can be easily extended to the case in which the nonnegative constraints are expressed in terms of an elliptic cone. One could even consider nonnegative constraints expressed in terms of a Cartesian product of several elliptic cones. LEMs with such sort of nonnegative constraints arise for instance in the analysis of elastic systems with unilateral contact and friction.

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