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Links between directional derivatives through multidirectional mean value inequalities

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Abstract We prove in the general setting of lower semicontinuous functions on Banach spaces the relation between the Rockafellar directional derivative and the mixed lower limit of the lower Dini derivatives. As a byproduct we derive the famous inclusions of tangent cones of closed sets in Banach spaces. The results are established using as principal tool multidirectional mean value inequalities [Aussel et al., SIAM J Optim **9**(3), 690–706 (1999)].

Keywords Directional derivatives · Tangent cones · Multidirectional mean value inequalities · Compactly epi-Lipschitzian function

Mathematics Subject Classification (2000) 26A24 · 49J52

To Alfred Auslender, on the occasion of his 65th birthday and of his award of a Dhc.

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1 Introduction

In the literature of the last 25 years we can find several definitions of directional derivatives stimulated by optimization problems and studies of properties of several classes of nonsmooth functions. Along with those definitions, naturally arises the interest to investigate the links that there are between these concepts of directional derivatives. Our aim is to study such links through a multidirectional mean value inequality established in Aussel et al. [2]. Such a mean value inequality involving sets has been provided for the first time by Clarke and Ledyaev [8] in their seminal paper relative to the Hilbert setting. Examples of the study of some particular relations between directional derivatives via a geometrical approach (i.e., via relations between tangent cones) exist in the literature. We can cite Rockafellar [21] who introduced (see Definition 1) the generalized directional derivative $f^{\uparrow}$ (that we denote in the paper by $d^{\uparrow} f$ in order to homogenize the notation) and established, concerning the Clarke derivative $d^0 f$ (denoted by f^0 in [21]), the relation

$$d^{\uparrow} f(\bar{x}; \bar{v}) = \liminf_{v \to \bar{v}} d^0 f(\bar{x}; v), \tag{1}$$

for directionally Lipschitzian functions. Also, Ioffe [14] showed that in finite dimensional spaces for every lower semicontinuous (lsc) function f, the generalized directional derivative $d^{\uparrow} f(\bar{x}; \cdot)$ is the upper epi-limit (or a Γ-limit) of the (lower) Dini directional derivative $d^- f(x; \cdot)$ when $x \to_f \bar{x}$, that is,

$$\lim \sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^- f(x; v) = d^{\uparrow} f(\bar{x}; \bar{v}). \tag{2}$$

Among others, our interest for the study of the above links is that equalities like (1) and (2) may be used for example to investigate properties of the Rockafellar derivative of a value function in sensitivity analysis. Observe that other theoretical applications of (1) and (2) are given in [14], [20] and [21].

In the early 80s, the tangent cone introduced by Clarke in 1973 focussed much attention because of its relevance in variational and nonsmooth analysis due, in particular, to its convexity property. On the other hand, the theory of approximating cones originated by Boulingand [7] was strongly revisited in the same period. The Boulingand tangent cone always contains the Clarke one. Motivated, among others by the viability theory of differential inclusions, Cornet initiated in the finite dimensional setting the study of the lower limit of Boulingand tangent cones. Actually, Cornet [10] and Penot [18] proved that this lower limit is equal, in finite dimensions, to the Clarke tangent cone. Penot's approach also yields to the equality for some classes of sets of Banach space. The general inclusion of the lower limit of Boulingand tangent cones in the Clarke tangent cone in the context of Banach space has been established by Treiman [24].

Links between directional derivatives and the theory of approximating tangent cones are strongly related. In fact, Ioffe in [14] used the result given by Cornet and Penot in order to prove the above equality (2). Also, the proof of (1) given by Rockafellar in

[21] is in terms of tangential cones where the epigraph of a function is the involved set. The proof of (1) and (2) given in [22] also follows the same geometrical approach.

In this work, as we already said, our objective is to obtain links between directional derivatives with a functional approach, i.e., not using inclusions between tangent and approximated tangent cones. The principal tool for this purpose is a multidirectional mean value inequality given by Aussel et al. in [2] for a large class of subdifferential operators and another corresponding multidirectional mean value inequality established for the Dini directional derivative, both based on the Ekeland's variational principle. The result in [2] is an extension to Banach space of the one of Clarke and Ledyaev [8] established earlier in the context of Hilbert space (see also [9] for applications). Another multidirectional mean value inequality has been also provided by Luc [17]. With these results we can recover the well known inclusions and equalities from the theory of approximating tangent cones cited above, setting as lsc function the indicator function of a closed set. Also, we obtain the results given by Borwein and Strójwas [6] for weak sequential Bouligand tangent cones in reflexive Banach spaces as corollary of links that involve the weak lower Dini directional derivative. So, we provide some more theoretical applications showing how strong is the multidirectional mean value inequality in [2].

The paper is organized as follows. In Sect. 2 we recall definitions of some nonsmooth analysis concepts. The third section is devoted to establishing the inequality

$$d^{\uparrow} f(\bar{x}; \bar{v}) \leq \limsup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^{-} f(x; v), \tag{3}$$

on any Banach space for any lsc function and also the same inequality with the weak lower Dini directional derivative instead of $d^{-} f$. Finally, in Sect. 4 we investigate classes of functions for which one has the equality in (3). For this purpose, we introduce a new directional derivative which is related to the tangent cone defined by Borwein and Strójwas in [5].

2 Preliminaries

Throughout all the paper, unless othewise stated, $(X, \|\cdot\|)$ stands for a real Banach space, X^* for its topological dual and $\langle \cdot, \cdot \rangle$ for the duality pairing.

We recall some well known concepts in nonsmooth analysis, as lower Dini, weak lower Dini, and Rockafellar directional derivatives. They are special limits of the differential quotient $t^{-1}[f(x + tv) - f(x)]$. We recall first the concept of upper epi-limit or Γ-limit (see [1,13,22]). Let $\varphi : X \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lsc function. For $\bar{x}, \bar{v} \in X$ the upper epi-limit or Γ-limit of $\varphi(x; \cdot)$ at $\bar{v}$ when $x \to_f \bar{x}$ is the mixed limit

$$\begin{aligned}\limsup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} \varphi(x; v) &:= \sup_{\delta > 0} \inf_{\varepsilon > 0} \sup_{x \in B_f(\bar{x}, \varepsilon)} \inf_{v \in B(\bar{v}, \delta)} \varphi(x; v) \\ &= \sup_{x_k \to_f \bar{x}} \inf_{v_k \to \bar{v}} \limsup_{k \to +\infty} \varphi(x_k; v_k),\end{aligned}$$

where $B(\bar{v},\delta) = \{v \in X \ : \ \|\bar{v} - v\| < \delta\}$ is the open ball centered in $\bar{v} \in X$ with radius $\delta > 0$, $B_f(\bar{x},\varepsilon) = B(\bar{x},\varepsilon) \cap \{x \in X \ : \ |f(\bar{x}) - f(x)| < \varepsilon\}$, and $x_k \to_f \bar{x}$ means $x_k \to \bar{x}$ with $f(x_k) \to f(\bar{x})$.

Note that the above definition does not change if we put the closed ball $B[\bar{v},\delta] := \{v \in X \ : \ \|\bar{v} - v\| \leq \delta\}$ instead of $B(\bar{v},\delta)$.

Definition 1 The lower Dini directional derivative of a lsc function f at a point $\bar{x}$ where it is finite, is defined by

$$d^- f(\bar{x};\bar{v}) = \liminf_{\substack{v\to\bar{v}\\ t\to 0^+}} t^{-1}[f(\bar{x}+tv) - f(\bar{x})];$$

the weak lower Dini directional derivative by the sequential lower limit, with the weak topology in the notation $v_k \rightharpoonup \bar{v}$,

$$d^-_w f(\bar{x};\bar{v}) = \inf_{\substack{v_k\rightharpoonup\bar{v}\\ t_k\to 0}} \liminf_{k\to\infty} t_k^{-1}[f(\bar{x}+t_k v_k) - f(\bar{x})];$$

and the Rockafellar directional derivative by the following upper epi-limit or Γ-limit of the differential quotient

$$\begin{aligned} d^\uparrow f(\bar{x};\bar{v}) &= \lim\sup_{\substack{x\to_f\bar{x}\\ t\to 0^+}} \inf_{v\to\bar{v}} t^{-1}[f(x+tv) - f(x)] \\ &:= \sup_{\delta>0}\inf_{\varepsilon>0} \sup_{\substack{x\in B_f(\bar{x},\varepsilon)\\ t\in]0,\varepsilon[}} \inf_{v\in B(\bar{v},\delta)} t^{-1}[f(x+tv) - f(x)] \\ &= \sup_{\substack{x_k\to_f\bar{x}\\ t_k\to 0^+}} \inf_{v_k\to\bar{v}} \limsup_{k\to+\infty} t_k^{-1}[f(x_k+t_k v_k) - f(x_k)]. \end{aligned}$$

Obviously, the definition of $d^\uparrow f(\bar{x};\bar{v})$ does not change if we put the closed ball $B[\bar{v},\delta]$ instead of $B(\bar{v},\delta)$.

Also, we recall the definitions of Bouligand, weak sequential Bouligand, and Clarke tangent cones.

Definition 2 Let $S \subset X$ be a nonempty closed set. The Bouligand tangent cone to S at $x \in S$ is defined by

$$\begin{aligned} T_B(S;x) &:= \limsup_{t\to 0^+} t^{-1}(S-x) \\ &= \left\{v \in X \ : \ \exists\, t_k \to 0^+,\ \exists\, v_k \to v \text{ with } v_k \in t_k^{-1}(S-x)\right\}; \end{aligned}$$

the weak sequential Boulingand tangent cone to S at $x \in S$ by

$$\begin{aligned} T_B^w(S;x) &:= w - \limsup_{t\to 0^+} t^{-1}(S-x) \\ &= \left\{v \in X \ : \ \exists\, t_k \to 0^+,\ \exists\, v_k \rightharpoonup v \text{ with } \ v_k \in t_k^{-1}(S-x)\right\}, \end{aligned}$$

where $v_k \rightharpoonup v$ stands (as in Definition 1) for the convergence with respect to the weak topology; and the Clarke tangent cone to S at $x \in S$ by

$$\begin{aligned} T_C(S;x) &:= \liminf_{\substack{x' \to_S x \\ t \to 0^+}} t^{-1}(S - x') \\ &= \left\{ v \in X \ : \ \forall\, x_k \to_S x,\ \forall\, t_k \to 0^+,\ \exists\, v_k \to v \text{ with } v_k \in t_k^{-1}(S - x_k) \right\}, \end{aligned}$$

where $x_k \to_S \bar{x}$ means $x_k \to \bar{x}$ with $x_k \in S$.

Directly from the above definitions, we can observe that for $\bar{x} \in S$ and for all $\bar{v} \in X$ one has (see also [14,21])

$$\Psi_{T_B(S;\bar{x})}(\bar{v}) = d^- \Psi_S(\bar{x}; \bar{v}) \tag{4}$$
$$\Psi_{T_B^w(S;\bar{x})}(\bar{v}) = d_w^- \Psi_S(\bar{x}; \bar{v}) \tag{5}$$
$$\Psi_{T_C(S;\bar{x})}(\bar{v}) = d^{\uparrow} \Psi_S(\bar{x}; \bar{v}) \tag{6}$$

where

$$\Psi_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S, \end{cases}$$

is the indicator function of the set S.

On the other hand, recalling that the epigraph of f is given by

$$\text{epi } f := \{(x, r) \in X \times \mathbb{R} \ : \ f(x) \le r\},$$

we see directly that for all $\bar{x} \in \text{dom } f := \{x \in X \ : \ f(x) < +\infty\}$, and $(\bar{v}, s) \in X \times \mathbb{R}$ one has

$$d^- \Psi_{\text{epi } f}(\bar{x}, f(\bar{x}); \bar{v}, s) = \Psi_{\text{epi } d^- f(\bar{x};\cdot)}(\bar{v}, s); \tag{7}$$
$$d_w^- \Psi_{\text{epi } f}(\bar{x}, f(\bar{x}); \bar{v}, s) = \Psi_{\text{epi } d_w^- f(\bar{x};\cdot)}(\bar{v}, s); \tag{8}$$
$$d^{\uparrow} \Psi_{\text{epi } f}(\bar{x}, f(\bar{x}); \bar{v}, s) = \Psi_{\text{epi } d^{\uparrow} f(\bar{x};\cdot)}(\bar{v}, s); \tag{9}$$
$$f(\bar{x}) \le \beta \Rightarrow \begin{cases} \Psi_{\text{epi } f}(\bar{x} + \bar{v}, \beta + s) \le \Psi_{\text{epi } f}(\bar{x} + \bar{v}, f(\bar{x}) + s) \\ \\ d^- \Psi_{\text{epi } f}(\bar{x}, \beta; \bar{v}, s) \le d^- \Psi_{\text{epi } f}(\bar{x}, f(\bar{x}); \bar{v}, s). \end{cases} \tag{10}$$

Finally, following [12,23] we will call *quasi presubdifferential* on X any operator ∂ which associates with any function $f : X \to \mathbb{R} \cup \{+\infty\}$ and any $x \in X$ a subset $\partial f(x)$ of X^* and which satisfies the following properties.

- (P1) $\partial f(x) \subset X^*$ and $\partial f(x) = \emptyset$ if $x \notin \text{dom } f$;
- (P2) $\partial f(x) = \partial g(x)$ whenever f and g coincide on a neighborhood of x;
- (P3) $\partial f(x)$ is equal to the subdifferential in the sense of convex analysis whenever f is convex and lsc;

(P4) If f is lsc near x, g convex continuous, and x a local minimum point of $f+g$, one has

$$0 \in (\limsup_{y \to_f x} \partial f(y)) + \partial g(x),$$

where lim sup denotes here the weak-star sequential upper limit.

3 General inequalities between directional derivatives

In this section we show links between directional derivatives which are established for any lsc function on any Banach space. In order to obtain these links, we need the multidirectional mean value inequalities below. The first part is established in [2] for an operator ∂ which is a quasi presubdifferential, and the second result is given in terms of the Dini's derivative which is obtained following an analogous proof to that of [2].

Let $C \subset X$ be a nonempty convex set. For $\delta \geq 0$, we set $B_\delta(C) := \{x \in X : \mathrm{d}_C(x) < \delta\}$, where $\mathrm{d}_C(\cdot)$ is the distance function to C. As in [2], we set $r_C(f) := \sup_{\delta > 0} \inf_{y \in B_\delta(C)} f(y)$.

Theorem 1 *Let $C \subset X$ be a nonempty convex closed set, $a \in X$, $D = [a, C] := \{\lambda a + (1-\lambda)c \ : \ \lambda \in [0,1], \ \ c \in C\}$, and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lsc function bounded from below on a neighborhood of D. If $a \in \operatorname{dom} f$ and $r \leq r_C(f)$, then:*

(a) *For any quasi presubdifferential ∂, there exist sequences $\{x_n\} \subset \operatorname{Dom} \partial f$ and $x_n^* \in \partial f(x_n)$ such that $\mathrm{d}_D(x_n) \to 0$ and*

$$\langle x_n^*, c - a \rangle \geq r - f(a) - (1/n)\|c - a\| - 1/n \quad \forall c \in C, \ \forall n. \tag{11}$$

(b) *There exists a sequence $\{x_n\} \subset \operatorname{dom} f$ such that $\mathrm{d}_D(x_n) \to 0$ and*

$$d^- f(x_n; c - a) \geq r - f(a) - (1/n)\|c - a\| - 1/n \quad \forall c \in C, \ \forall n. \tag{12}$$

Proof One can find the proof of (11) in [2]. The relation (12) can be established following the proof of (11) in [2].

As a consequence of the above theorem we have the next proposition. Recall first that a Banach space is an Asplund space provided every separable subspace has a separable topological dual.

Proposition 1 *Let $C \subset X$ be a nonempty convex closed set of an Asplund space X, $a \in X$, $D = [a, C]$, and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ a lsc function bounded from below on a neighborhood of D. If $a \in \operatorname{dom} f$ and $r \leq r_C(f)$, then there exists a sequence $\{x_n\} \subset \operatorname{dom} f$ such that $\mathrm{d}_D(x_n) \to 0$ and*

$$d_w^- f(x_n; c - a) \geq r - f(a) - (1/n)\|c - a\| - 1/n \quad \forall c \in C, \ \forall n. \tag{13}$$

In order to prove this result, we need the following lemma.

Lemma 1 *For all* $\bar{x} \in \operatorname{Dom} \partial_F f = \{x \in X \ : \ \partial_F f(x) \neq \emptyset\}$, *one has*

$$x^* \in \partial_F f(\bar{x}) \Rightarrow \langle x^*, \bar{v}\rangle \leq d_w^- f(\bar{x}; \bar{v}) \quad \forall\, \bar{v} \in X,$$

where $\partial_F f(x) := \left\{x^* \in X^* \ : \ \liminf_{x' \to x} \frac{f(x') - f(x) - \langle x^*, x' - x\rangle}{\|x' - x\|} \geq 0\right\}$ *is the Fréchet subdifferential of the function* f.

Proof (Lemma) Let $\bar{v}$ be in X and $x^* \in \partial_F f(\bar{x})$. Take any $t_k \to 0^+$ and (v_k) converging weakly to $\bar{v}$. For every $\varepsilon > 0$, there exists $k_0 \in I\!N$ such that

$$t_k^{-1}[f(\bar{x} + t_k v_k) - f(\bar{x})] \geq \langle x^*, v_k\rangle + \varepsilon \|v_k\| \quad \forall\, k \geq k_0,$$

hence according to the weak lower semicontinuity of the norm

$$\liminf_{k \to +\infty} t_k^{-1}[f(\bar{x} + t_k v_k) - f(\bar{x})] \geq \langle x^*, v\rangle + \varepsilon \|\bar{v}\|.$$

We then deduce $d_w^- f(\bar{x}; \bar{v}) \geq \langle x^*, \bar{v}\rangle + \varepsilon\|\bar{v}\|$ for all $\varepsilon > 0$, which proves the lemma.

Proof (Proposition 1) In Asplund spaces, the Fréchet subdifferential ∂_F is a quasi presubdifferential (see for example [12]). Moreover, from Lemma 1, $x_n^* \in \partial_F f(x_n) \Rightarrow \langle x_n^*, c - a\rangle \leq d_w^- f(x_n; c - a)$. Replacing in (11) we obtain the desired result.

We establish now the following first inequality.

Theorem 2 *For any lsc function* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *one has*

$$d^{\uparrow} f(\bar{x}; \bar{v}) \leq \lim \sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^- f(x; v) \qquad \forall\, \bar{x} \in \operatorname{dom} f,\ \bar{v} \in X. \tag{14}$$

Proof We will develop this proof in two steps. First, we suppose that f is continuous with respect to its domain in order to obtain (14), and then using this result we will extend (14) for every lsc function.

Step 1. Fix $\bar{x} \in \operatorname{dom} f$ and $\bar{v} \in X$. By continuity with respect to the domain of f at $\bar{x} \in \operatorname{dom} f$, we have

$$\forall\, \eta > 0, \ \ \exists\, \xi = \xi(\eta) > 0 \ \text{ such that } \ x \in B(\bar{x}, \xi) \cap \operatorname{dom} f \Rightarrow x \in B_f(\bar{x}, \eta). \tag{15}$$

Also, since f is lsc at $\bar{x}$ we have that there exists $\bar{\nu} > 0$ such that f is bounded from below on $B(\bar{x}, \bar{\nu})$. Let $\delta, \eta > 0$, and $\varepsilon > 0$ such that

$$\varepsilon(2 + \delta + \|\bar{v}\|) < \min\{\xi(\eta), \bar{\nu}\}$$

where $\xi(\eta)$ is given by (15), and let $\alpha > 0$ and $\delta' \in]0, \delta[$.

Let $x \in B_f(\bar{x}, \varepsilon)$ and $t \in]0, \varepsilon[$. Put $C := x + tB[\bar{v}, \delta']$. Note that f is bounded from below on a neighborhood of $D := [x, C]$ because $[x, C] \subset B(\bar{x}, \bar{\nu})$.

Choose $\gamma = \gamma(t, \delta', \delta) > 0$ such that $\delta' + \frac{\gamma}{t} < \delta$. Since $r_C(f) = \sup_{\gamma'>0} \inf_{B_{\gamma'}(C)} f$, we have $\inf_{B_\gamma(C)} f < r_C(f) + t\alpha$, which yields the existence of some $c \in C$ and $b \in B[0, 1]$ with $f(c+\gamma b) < r_C(f)+t\alpha$. Thus, we have some $v'' \in B[\bar{v}, \delta' + \frac{\gamma}{t}] \subset B[\bar{v}, \delta]$ such that $f(x + tv'') < r_C(f) + t\alpha$. By Theorem 1 (12), there exists $x_n = x_n(x, t, \delta, \varepsilon)$ in dom f such that $d_D(x_n) \to 0$ and

$$d^- f(x_n; v') \geq t^{-1}\big[f\big(x + tv''\big) - f(x)\big] - \alpha - (1/n)\|v'\| - \frac{1}{tn} \quad \forall\, v' \in B[\bar{v}, \delta']$$

yielding for $\beta := \|\bar{v}\| + \delta$

$$\inf_{v' \in B[\bar{v},\delta']} d^- f\big(x_n; v'\big) \geq t^{-1}\big[f\big(x + tv''\big) - f(x)\big] - \alpha - \frac{\beta}{n} - \frac{1}{tn}$$

and hence

$$\inf_{v' \in B[\bar{v},\delta']} d^- f\big(x_n; v'\big) \geq \inf_{v \in B[\bar{v},\delta]} t^{-1}[f(x + tv) - f(x)] - \alpha - \frac{\beta}{n} - \frac{1}{tn}.$$

Choose $n_0 = n_0(x, t, \delta, \varepsilon)$ such that for all $n \geq n_0$ one has $\mathrm{d}_D(x_n) < \varepsilon$ and hence there exist $\alpha_n \in [0, 1]$ and $v_n \in B[\bar{v}, \delta]$ such that $\|x_n - d_n\| < \varepsilon$ for $d_n := x + t\alpha_n v_n$. Thus, for each $n \geq n_0$

$$\|x_n - \bar{x}\| \leq \|x_n - d_n\| + \|d_n - \bar{x}\| < \varepsilon + \|\bar{x} - x\| + t\|v_n\| < 2\varepsilon + \varepsilon(\|\bar{v}\| + \delta) < \xi,$$

ensuring $x_n \in B_f(\bar{x}, \eta)$ according to (15) and hence

$$\inf_{v \in B[\bar{v},\delta]} t^{-1}[f(x + tv) - f(x)] \leq \sup_{x' \in B_f(\bar{x},\eta)} \inf_{v' \in B[\bar{v},\delta']} d^- f\big(x'; v'\big) + \alpha + \frac{\beta}{n} + \frac{1}{tn}.$$

This entails for all $x \in B_f(\bar{x}, \varepsilon),\ t \in]0, \varepsilon[$

$$\inf_{v \in B[\bar{v},\delta]} t^{-1}[f(x + tv) - f(x)] \leq \sup_{x' \in B_f(\bar{x},\eta)} \inf_{v' \in B[\bar{v},\delta']} d^- f\big(x'; v'\big) + \alpha.$$

Therefore, we have successively

$$\sup_{\substack{x \in B_f(\bar{x},\varepsilon) \\ t \in]0,\varepsilon[}} \inf_{v \in B[\bar{v},\delta]} t^{-1}[f(x + tv) - f(x)] \leq \sup_{x' \in B_f(\bar{x},\eta)} \inf_{v' \in B[\bar{v},\delta']} d^- f\big(x'; v'\big) + \alpha$$

$$\inf_{\varepsilon>0} \sup_{\substack{x \in B_f(\bar{x},\varepsilon) \\ t \in]0,\varepsilon[}} \inf_{v \in B[\bar{v},\delta]} t^{-1}[f(x + tv) - f(x)] \leq \sup_{x' \in B_f(\bar{x},\eta)} \inf_{v' \in B[\bar{v},\delta']} d^- f\big(x'; v'\big) + \alpha$$

with $\eta > 0$ arbitrary, so

$$\begin{aligned}&\inf_{\varepsilon>0} \sup_{\substack{x\in B_f(\bar{x},\varepsilon)\\ t\in]0,\varepsilon[}} \inf_{v\in B[\bar{v},\delta]} t^{-1}[f(x+tv)-f(x)]\\ &\quad\le \sup_{\delta''>0} \inf_{\eta>0} \sup_{x'\in B_f(\bar{x},\eta)} \inf_{v'\in B[\bar{v},\delta'']} d^- f\left(x'; v'\right)+\alpha\end{aligned}$$

for any $\delta,\ \alpha > 0$, which proves (14).

Step 2. Assume now that f is lsc. We may suppose that the second member of (14) is less than $+\infty$. Fix any $r \in \mathbb{R}$ such that

$$r > \lim \sup_{x\to_f \bar{x}} \inf_{v\to\bar{v}} d^- f(x; v) = \sup_{x_k\to_f \bar{x}} \inf_{v_k\to\bar{v}} \limsup_{k\to+\infty} d^- f(x_k; v_k). \tag{16}$$

Take $x_k \to \bar{x}$ and $s_k \ge f(x_k)$ such that $s_k \to f(\bar{x})$. Since f is lsc, one has $f(x_k) \to f(\bar{x})$ and then $x_k \to_f \bar{x}$. From (16) one knows that there exists $v_k \to \bar{v}$ such that $d^- f(x_k; v_k) < r$ for all k large enough. Using (7) we obtain that $d^-\Psi_{\text{epi}\, f}(x_k, f(x_k); v_k, r) = 0$, and from (10) we can deduce $d^-\Psi_{\text{epi}\, f}(x_k, s_k; v_k, r) = 0$. Thus, for all $(x_k, s_k) \to_{\text{epi}\, f} (\bar{x}, f(\bar{x}))$ there exists $v_k \to \bar{v}$ such that

$$\limsup_{k\to+\infty} d^-\Psi_{\text{epi}\, f}(x_k, s_k; v_k, r) = 0,$$

which implies

$$\begin{aligned}0 &= \sup_{(x_k,s_k)\to_{\text{epi}\, f}(\bar{x},f(\bar{x}))} \inf_{(v_k,r_k)\to(\bar{v},r)} \limsup_{k\to+\infty} d^-\Psi_{\text{epi}\, f}(x_k, s_k; v_k, r_k)\\ &= \lim \sup_{(x,s)\to_{\text{epi}\, f}(\bar{x},f(\bar{x}))} \inf_{(v',r')\to(\bar{v},r)} d^-\Psi_{\text{epi}\, f}(x, s; v', r').\end{aligned}$$

Since $\Psi_{\text{epi}\, f}$ is continuous with respect to its domain, we obtain by the first step $d^{\uparrow}\Psi_{\text{epi}\, f}(\bar{x}, f(\bar{x}); \bar{v}, r) = 0$ and using (9), we conclude $d^{\uparrow} f(\bar{x}; \bar{v}) \le r$, which proves the desired result.

Using Proposition 1 and following the proof of Theorem 2, we obtain the following theorem giving a similar inequality with the weak sequential lower Dini derivative.

Theorem 3 *If X is an Asplund space then, for all lsc function $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ one has*

$$d^{\uparrow} f(\bar{x}; \bar{v}) \le \lim \sup_{x\to_f \bar{x}} \inf_{v\to\bar{v}} d_w^- f(x; v) \qquad \forall\, \bar{x} \in \operatorname{dom} f,\ \bar{v} \in X. \tag{17}$$

Assume now that the following additional properties hold for the quasi presubdifferential ∂.

(P5) For any lsc functions $f : X \to \mathbb{R} \cup \{+\infty\}$, $g : Y \to \mathbb{R} \cup \{+\infty\}$ (where Y is another Banach space), and for $(f \oplus g)(x, y) := f(x) + g(y)$ one has $\partial(f \oplus g)(x, y) \subset \partial f(x) \times \partial g(y)$;
(P6) For any lsc function f and any $(x^*, -r^*) \in \partial\Psi_{\mathrm{epi}\, f}(x, r)$ one has $r^* \geq 0$;
(P7) $\partial\Psi_{\mathrm{epi}\, f}(x, r) \subset \partial\Psi_{\mathrm{epi}\, f}(x, f(x))$ for any lsc function f and any $(x, r) \in X \times \mathbb{R}$;
(P8) For any lsc function f for any $x \in \mathrm{dom}\, f$ it is satisfied $\partial f(x) = \big\{x^* \in X^* : (x^*, -1) \in \partial\Psi_{\mathrm{epi}\, f}(x, f(x))\big\}$.

Then for a lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$, if we denote by $\partial^\infty f(x)$ its related singular presubdifferential at x, i.e.,

$$\partial^\infty f(x) := \big\{x^* \in X^* : (x^*, 0) \in \partial\Psi_{\mathrm{epi}\, f}(x, f(x))\big\},$$

and if we define

$$\sigma_f(x; v) := \sup\big\{\langle x^*, v\rangle : x^* \in \partial f(x) \cup \partial^\infty f(x)\big\},$$

we can obtain another result similar to Theorem 2.

Observe that smaller is the quasi presubdifferential better is the function σ_f with respect to the inequality (18) below. All the properties (P1)–(P8) hold in appropriate spaces for the proximal, Fréchet, and viscosity subdifferentials which are known as the smallest ones.

Theorem 4 *For any lsc function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, if ∂ is a quasi presubdifferential that satisfies properties* (P5)–(P8)*, one has*

$$d^\uparrow f(\bar{x}; \bar{v}) \leq \lim \sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} \sigma_f(x; v) \qquad \forall\, \bar{x} \in \mathrm{dom}\, f,\ \bar{v} \in X. \tag{18}$$

Proof Following the proof of Theorem 2, we see that the arguments of **Step 1** (the lsc function is continuous with respect to its domain) are the same using the corresponding multidirectional mean value inequality (11) from Theorem 1. Thus, we will only prove the **Step 2**. Firstly, let us prove that for $x \in \mathrm{dom}\, f$ and $r \geq f(x)$ one has

$$\sigma_{\Psi_{\mathrm{epi}\, f}}(x, r; v, s) \leq \sigma_{\Psi_{\mathrm{epi}\, f}}(x, f(x); v, s) \leq \Psi_{\mathrm{epi}\, \sigma_f(x,\cdot)}(v, s) \qquad \forall\, (v, s) \in X \times \mathbb{R}. \tag{19}$$

If we fix $x \in \mathrm{dom}\, f$ and $r \geq f(x)$, we can show that

$$\partial^\infty \Psi_{\mathrm{epi}\, f}(x, r) \subset \partial\Psi_{\mathrm{epi}\, f}(x, r). \tag{20}$$

In fact, put $g := \Psi_{\mathrm{epi}\, f}$ and take any $(x^*, -r^*) \in \partial^\infty g(x, r)$. This means that $(x^*, -r^*, 0) \in \partial\Psi_{\mathrm{epi}\, g}(x, r, 0)$. As $\Psi_{\mathrm{epi}\, g}(x, r, s) = \Psi_{\mathrm{epi}\, f}(x, r) + \Psi_{[0,+\infty[}(s)$, according to (P3) and (P5) we have $\partial\Psi_{\mathrm{epi}\, g}(x, r, 0) \subset \partial\Psi_{\mathrm{epi}\, f}(x, r) \times]-\infty, 0]$ and hence $(x^*, -r^*) \in \partial\Psi_{\mathrm{epi}\, f}(x, r)$, which yields the desired inclusion.

Fix now any $(v, s) \in X \times \mathbb{R}$. Inclusion (20) and property (P7) yield

$$\partial \Psi_{\text{epi}\, f}(x, r) \cup \partial^\infty \Psi_{\text{epi}\, f}(x, r) = \partial \Psi_{\text{epi}\, f}(x, r) \subset \partial \Psi_{\text{epi}\, f}(x, f(x)), \tag{21}$$

which, in particular, entails $\sigma_{\Psi_{\text{epi}\, f}}(x, r; v, s) \leq \sigma_{\Psi_{\text{epi}\, f}}(x, f(x); v, s)$, i.e., the first inequality in (19).

Let us establish the second inequality in (19). It is enough to suppose that $\Psi_{\text{epi}\, \sigma_f(x,\cdot)}(v, s) = 0$, i.e., $\sigma_f(x, v) \leq s$. Take $(x^*, -r^*) \in \partial \Psi_{\text{epi}\, f}(x, f(x)) \cup \partial^\infty \Psi_{\text{epi}\, f}(x, f(x))$. By the equality in (21) with $r = f(x)$, we have $(x^*, -r^*) \in \partial \Psi_{\text{epi}\, f}(x, f(x))$ and hence $r^* \geq 0$ according to (P6).

If $r^* > 0$ then, by (P8), we have $x^*/r^* \in \partial f(x)$ and we obtain $\langle x^*/r^*, v\rangle \leq s$, i.e., $\langle x^*, v\rangle - r^* s \leq 0$.

On the other hand, if $r^* = 0$ we have $x^* \in \partial^\infty f(x)$ and then $\langle x^*, v\rangle \leq s$. With this inequality and recalling that $\partial^\infty f(x)$ is a cone, we conclude $\langle x^*, v\rangle \leq 0$. Thus, for all $(x^*, -r^*) \in \partial \Psi_{\text{epi}\, f}(x, f(x)) \cup \partial^\infty \Psi_{\text{epi}\, f}(x, f(x))$ we have proved that

$$\langle x^*, v\rangle - r^* s \leq 0.$$

This means that the second inequality in (19) holds.

Now we prove the **Step 2** (the function f is lsc). We may suppose that the second member of (18) is less than $+\infty$. Fix any $r \in \mathbb{R}$ such that

$$r > \limsup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} \sigma_f(x; v) = \sup_{x_k \to_f \bar{x}} \inf_{v_k \to \bar{v}} \limsup_{k \to +\infty} \sigma_f(x_k; v_k). \tag{22}$$

Take $x_k \to \bar{x}$ and $s_k \geq f(x_k)$ such that $s_k \to f(\bar{x})$. Since f is lsc, one has $f(x_k) \to f(\bar{x})$ and then $x_k \to_f \bar{x}$. From (22) one knows that there exists $v_k \to \bar{v}$ such that $\sigma_f(x_k; v_k) < r$ for all k large enough. Using (19) we obtain that $\sigma_{\Psi_{\text{epi}\, f}}(x_k, s_k; v_k, r) = 0$.

Thus, for all $(x_k, s_k) \to_{\text{epi}\, f} (\bar{x}, f(\bar{x}))$ there exists $v_k \to \bar{v}$ such that

$$\limsup_{k \to +\infty} \sigma_{\Psi_{\text{epi}\, f}}(x_k, s_k; v_k, r) = 0,$$

which implies

$$\begin{aligned} 0 &= \sup_{(x_k, s_k) \to_{\text{epi}\, f} (\bar{x}, f(\bar{x}))} \inf_{(v_k, r_k) \to (\bar{v}, r)} \limsup_{k \to +\infty} \sigma_{\Psi_{\text{epi}\, f}}(x_k, s_k; v_k, r_k) \\ &= \lim \sup_{(x, s) \to_{\text{epi}\, f} (\bar{x}, f(\bar{x}))} \inf_{(v', r') \to (\bar{v}, r)} \sigma_{\Psi_{\text{epi}\, f}}\left(x, s; v', r'\right). \end{aligned}$$

Since $\Psi_{\text{epi}\, f}$ is continuous with respect to its domain, we obtain by the first step $d^\uparrow \Psi_{\text{epi}\, f}(\bar{x}, f(\bar{x}); \bar{v}, r) = 0$ and using (9), we conclude $d^\uparrow f(\bar{x}; \bar{v}) \leq r$, which proves the desired result.

The links (14) and (17) established between directional derivatives and equalities (4)–(6) allow us to derive the following well known inclusions between tangent cones.

The first inclusion (23) below in the context of Banach spaces is due to Treiman [24]. His deep proof is based on the arguments of Bishop and Phelps [3]. Another proof has been given by Penot [19]. The inclusion (24) (as well as the equality) has been established for reflexive Banach spaces by Borwein and Strójwas [6]. The extension of this inclusion (24) in the setting of Asplund spaces has been obtained by Jourani [15].

Corollary 1 *Let $S \subset X$ be a nonempty closed set and $\bar{x} \in S$, then*:

$$\liminf_{x \to_S \bar{x}} T_B(S; x) \subset T_C(S; \bar{x}); \tag{23}$$

and if X is an Asplund space, one has

$$\liminf_{x \to_S \bar{x}} T_B^w(S; x) \subset T_C(S; \bar{x}). \tag{24}$$

Proof Let $\bar{v} \in \liminf_{x \to_S \bar{x}} T_B(S; x)$. Then for all $x_k \to_S \bar{x}$, there exists $v_k \to \bar{v}$ such that $v_k \in T_B(S; x_k)$, hence by (4), $\Psi_{T_B(S;x_k)}(v_k) = d^-\Psi_S(x_k; v_k) = 0$, that is,

$$\lim\sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^-\Psi_S(x; v) = \sup_{x_k \to_S \bar{x}} \inf_{v_k \to \bar{v}} \limsup_{k \to +\infty} d^-\Psi_S(x_k; v_k) = 0.$$

From Theorem 2 and using (6) we can write

$$0 = \lim\sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^-\Psi_S(x; v) \geq d^{\uparrow}\Psi_S(\bar{x}; \bar{v}) = \Psi_{T_C(S;\bar{x})}(\bar{v}) \geq 0,$$

which proves the inclusion (23).

The proof of (24) is analogous using Theorem 3.

In the next section we analyze some classes of functions for which we obtain the equality in (14) and (17). These results will imply equalities in (23) and (24) for corresponding classes of sets.

4 Special classes of functions

In this section we provide a detailed analysis of functions for which one has the equality in (14) and also in (17). Ioffe in [14] showed the equality in (14) in finite dimensional spaces. We obtain the same result and we established in addition that the equality holds for compactly epi-Lipschitzian functions (see [4,16]) and convex functions on any Banach space. Also, the equality in (17) is established for any lsc function in reflexive Banach spaces.

The result given by Ioffe [14] in the finite dimensional setting uses formula (23) with the equality (see [10,18]). The equality in (14) is obtained in [14] taking as a closed set the epigraph of a lsc function. In the next proposition we prove Ioffe's result by a direct functional approach. The technique used in the proof will also allow us to obtain the equality in (14) for general classes of functions.

Proposition 2 ([14]) *If the Banach space X is finite dimensional, then for any lsc function f and for all $\bar{x} \in \operatorname{dom} f$ and $\bar{v} \in X$ one has*

$$\lim_{x \to_f \bar{x}} \sup \inf_{v \to \bar{v}} d^- f(x; v) = d^{\uparrow} f(\bar{x}; \bar{v}). \tag{25}$$

Proof Fix $\bar{x} \in \operatorname{dom} f$, $\bar{v} \in X$, $\alpha > 0$, $\varepsilon > 0$, and $\delta > 0$. Then, for every $x \in B_f(\bar{x}, \varepsilon)$ and $v \in B[\bar{v}, \delta]$

$$\begin{cases} \text{there exist } \ \xi(x, v) > 0 \ \text{ such that} \\ d^- f(x; v) \le t^{-1}\left[f\left(x + tv'\right) - f(x)\right] + \alpha \\ \text{for all } \ t \in]0, \xi(x, v)] \ \text{ and } \ v' \in B(v, \xi(x, v)). \end{cases} \tag{26}$$

Since $B[\bar{v}, \delta]$ is compact,

$$\exists\, v_1, \ldots, v_n \ \text{ in } \ B[\bar{v}, \delta] \ \text{ such that } \ B[\bar{v}, \delta] \subset \bigcup_{i=1}^{n} B(v_i, \xi(x, v_i)). \tag{27}$$

If we take $\hat{v} \in \{v_i \ : \ i = 1, \ldots, n\}$ such that

$$d^- f(x; \hat{v}) = \min\{d^- f(x; v_i) \ : \ i = 1, \ldots, n\} \tag{28}$$

and $\eta = \min\{\varepsilon, \xi(x, v_1), \ldots, \xi(x, v_n)\}$, we obtain

$$\inf_{v \in B[\bar{v}, \delta]} d^- f(x; v) \le d^- f(x; \hat{v}) \le \alpha + \inf_{v' \in B[\bar{v}, \delta]} t^{-1}\left[f\left(x + tv'\right) - f(x)\right] \quad \forall\, t \in]0, \eta]$$

and hence

$$\sup_{x \in B_f(\bar{x}, \varepsilon)} \inf_{v \in B[\bar{v}, \delta]} d^-(x; v) \le \alpha + \sup_{\substack{x' \in B_f(\bar{x}, \varepsilon) \\ 0 < t \le \varepsilon}} \inf_{v' \in B[\bar{v}, \delta]} t^{-1}\left[f\left(x' + tv'\right) - f\left(x'\right)\right],$$

for $\varepsilon, \ \delta > 0$ and $\alpha > 0$ arbitrary. So the proof of the desired result is complete because the other inequality is established in Theorem 2.

In the case of a reflexive Banach space, the equality (25) holds with $d_w^-(\cdot; \cdot)$ in place of $d^-(\cdot; \cdot)$. However, the proof in this context requires, because of the sequential property in the definition of $d_w^-(\cdot; \cdot)$, some different arguments.

Proposition 3 *If the Banach space X is reflexive, then for any lsc function f and for all $\bar{x} \in \operatorname{dom} f$ and $\bar{v} \in X$ one has*

$$\lim_{x \to_f \bar{x}} \sup \inf_{v \to \bar{v}} d_w^- f(x; v) = d^{\uparrow} f(\bar{x}; \bar{v}).$$

Proof According to Theorem 3 we only need to prove that the first member of the proposition is not greater than the second one. Obviously we may suppose $d^{\uparrow} f(\bar{x}; \bar{v}) < +\infty$. Fix any real numbers $r > d^{\uparrow} f(\bar{x}; \bar{v})$ and $\delta > 0$. By definition of $d^{\uparrow} f(\bar{x}; \bar{v})$ there exist $\varepsilon_\delta > 0$ and $s_\delta > 0$ such that

$$\sup_{\substack{x \in B_f(\bar{x}, \varepsilon_\delta) \\ t \in]0, s_\delta[}} \inf_{v \in B[\bar{v}, \delta]} t^{-1}[f(x + tv) - f(x)] < r.$$

This yields for every $(t, x) \in]0, s_\delta[\times B_f(\bar{x}, \varepsilon_\delta)$ some $v(t, x) \in B[\bar{x}, \delta]$ with

$$t^{-1}[f(x + tv(t, x)) - f(x)] < r. \tag{29}$$

The weak compactness of $B[\bar{v}, \delta]$ allows us to fix some sequence $(t_n)_n$ in $]0, s_\delta[$ converging to 0 and such that $(v(t_n, x))_n$ converges weakly to some $v(x) \in B[\bar{v}, \delta]$. Combining (29) and the definition of $d_w^- f(x; \cdot)$ we obtain $d_w^- f(x; v(x)) \le r$ and hence $\inf_{v \in B[\bar{v}, \delta]} d_w^- f(x; v) \le r$. Consequently

$$\sup_{x \in B_f(x, \varepsilon_\delta)} \inf_{v \in B[\bar{v}, \delta]} d_w^- f(x; v) \le r$$

which yields

$$\sup_{\delta > 0} \inf_{\varepsilon > 0} \sup_{x \in B_f(x, \varepsilon)} \inf_{v \in B[\bar{v}, \delta]} d_w^- f(x; v) \le r$$

and completes the required inequality

$$\lim \sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d_w^- f(x; v) \le d^{\uparrow} f(\bar{x}; \bar{v}).$$

We proceed now to establish the equality (25) for directionally Lipschitzian functions. Recall that f is directionally Lipschitzian at $\bar{x}$ ([21]) provided there exists $\bar{z} \in X$ such that

$$f^{\square}(\bar{x}; \bar{z}) := \limsup_{\substack{x \to \bar{x} \\ t \to 0^+ \\ z \to \bar{z}}} t^{-1}[f(x + tz) - f(x)] < +\infty. \tag{30}$$

Our proof of (25) for directionally Lipschitzian functions will use the equality

$$d^{\uparrow} f(\bar{x}; \bar{v}) = \liminf_{v \to \bar{v}} d^0 f(\bar{x}; v), \tag{31}$$

which was proven by Rockafellar [21], where

$$d^0 f(\bar{x}; v) = \limsup_{\substack{x \to_f \bar{x} \\ t \to 0^+}} \frac{f(x + tv) - f(x)}{t}.$$

Rockafellar proved (31) when f is the indicator function of an epi-Lipschitzian closed set, so, he worked with sets and then, he concluded putting as epi-Lipschitzian closed set the epigraph of the directionally Lipschitzian lsc function (recall that a function is directionally Lipschitzian iff its epigraph is an epi-Lipschitzian set). In the proposition below, we give a direct analytical simple proof of (31).

Proposition 4 ([21]) *If the lsc function f is directionally Lipschitzian at $\bar{x} \in \operatorname{dom} f$, then for all $\bar{v} \in X$ one has*

$$d^{\uparrow} f(\bar{x}; \bar{v}) = \liminf_{v \to \bar{v}} d^0 f(\bar{x}; v).$$

Proof Directly from the definitions we have

$$d^{\uparrow} f(\bar{x}; \bar{v}) \leq \liminf_{v \to \bar{v}} d^0 f(\bar{x}; v),$$

for all $\bar{v} \in X$. Let us prove that

$$f^{\square}(\bar{x}; \bar{v} + \lambda \bar{w}) \leq d^{\uparrow} f(\bar{x}; \bar{v}) + \lambda f^{\square}(\bar{x}; \bar{w})$$

for all $\bar{v},\ \bar{w} \in X$ and $\lambda > 0$. Let $\bar{v} \in X$ such that $d^{\uparrow} f(\bar{x}; \bar{v}) < \infty$, $\bar{w} \in X$, $\lambda > 0$ and take $\varepsilon > 0$, $x_k \to_f \bar{x}$, $t_k \to 0^+$, and $q_k \to \bar{v} + \lambda \bar{w}$. As we will prove in Lemma 2, there exists $v_k \to \bar{v}$ such that $t_k^{-1}[f(x_k + t_k v_k) - f(x_k)] \leq d^{\uparrow} f(\bar{x}; \bar{v}) + \varepsilon$ with $x_k + t_k v_k \to_f \bar{x}$. Thus, we write

$$\begin{aligned} t_k^{-1}[f(x_k + t_k q_k) - f(x_k)] = t_k^{-1}&[f(x_k + t_k v_k) - f(x_k)] \\ &+ t_k^{-1}[f((x_k + t_k v_k) + t_k(q_k - v_k)) - f(x_k + t_k v_k)]. \end{aligned}$$

Since $x_k + t_k v_k \to_f \bar{x}$ and $(q_k - v_k) \to \lambda \bar{w}$, taking $\limsup_{k \to +\infty}$, we obtain

$$\begin{aligned} \limsup_{k \to +\infty} t_k^{-1}[f(x_k + t_k q_k) - f(x_k)] &\leq \limsup_{k \to +\infty} t_k^{-1}[f(x_k + t_k v_k) - f(x_k)] \\ &\quad + f^{\square}(\bar{x}; \lambda \bar{w}) \\ &\leq d^{\uparrow} f(\bar{x}; \bar{v}) + \varepsilon + \lambda f^{\square}(\bar{x}; \bar{w}), \end{aligned}$$

for all $\varepsilon > 0$, $x_k \to_f \bar{x}$, and $q_k \to \bar{v} + \lambda \bar{w}$. Thus,

$$f^{\square}(\bar{x}; \bar{v} + \lambda \bar{w}) \leq d^{\uparrow} f(\bar{x}; \bar{v}) + \lambda f^{\square}(\bar{x}; \bar{w}).$$

If in the above inequality we put $\bar{z}$ given by (30) in place of $\bar{w}$, one has

$$\liminf_{v \to \bar{v}} f^{\square}(\bar{x}; v) \leq \liminf_{\lambda \to 0^+} f^{\square}(\bar{x}; \bar{v} + \lambda \bar{z}) \leq d^{\uparrow} f(\bar{x}; \bar{v}),$$

which proves the result because $d^0 f \leq f^{\square}$.

We can now prove (25) for every directionally Lipschitzian lsc function. In the proof we use a result (see [11]) related to the upper semicontinuity of the function $d^0 f(\cdot; \bar{v})$ for all $\bar{v} \in X$.

Proposition 5 *If the lsc function* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is directionally Lipschitzian at* $\bar{x} \in \operatorname{dom} f$, *then*

$$\lim \sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^- f(x; v) = d^\uparrow f(\bar{x}; \bar{v}),$$

for all $\bar{v} \in X$.

Proof From [11] we know that for all $\bar{x} \in \operatorname{dom} f$ we have

$$\limsup_{x \to_f \bar{x}} d^0 f(x; v) \leq d^0 f(\bar{x}; v) \quad \text{for all } \; v \in X,$$

therefore

$$\limsup_{x \to_f \bar{x}} d^- f(x; v) \leq f^0(\bar{x}; v) \quad \text{for all } \; v \in X.$$

Taking the lower limit $\liminf_{v \to \bar{v}}$ and using Proposition 4 we obtain

$$\liminf_{v \to \bar{v}} \limsup_{x \to_f \bar{x}} d^- f(x; v) \leq d^\uparrow f(\bar{x}; \bar{v}),$$

which proves the result according to the inequality

$$\lim \sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^- f(x; v) \leq \liminf_{v \to \bar{v}} \limsup_{x \to_f \bar{x}} d^- f(x; v).$$

Another class of functions for which we have the equality (25) is the class of lsc convex functions.

Proposition 6 *If the lsc function* $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is convex, then*

$$\lim \sup_{x \to_f \bar{x}} \inf_{v \to \bar{v}} d^- f(x; v) = d^\uparrow f(\bar{x}; \bar{v}),$$

for all $\bar{x} \in \operatorname{dom} f$ *and* $\bar{v} \in X$.

Proof Fix $\bar{x} \in \operatorname{dom} f$ and $\bar{v} \in X$. Take $x_k \to_f \bar{x}$, $v_k \to \bar{v}$, and $\alpha > 0$. For every k, there exists $\delta_k = \delta_k(x_k, v_k) \leq 1/k$ such that

$$d^- f(x_k; v_k) \leq t^{-1}[f(x_k + tv'') - f(x_k)] + \alpha$$

for all $t \in]0, \delta_k]$ and $v'' \in B(v_k, \delta_k)$.

Since f is convex, we have

$$\delta_k^{-1}[f(x_k+\delta_k v_k)-f(x_k)]\le k[f(x_k+(1/k)v_k)-f(x_k)],$$

therefore $d^- f(x_k;v_k)\le \alpha + k[f(x_k+(1/k)v_k)-f(x_k)]$. Taking the upper limit $\limsup\limits_{k\to+\infty}$ and then the infimum $\inf\limits_{v_k\to\bar v}$, we obtain

$$\begin{aligned}\inf_{v_k\to\bar v}\limsup_{k\to+\infty} d^- f(x_k;v_k) &\le \alpha + \sup_{\substack{x_k'\to_f x\\ t_k\to 0^+}}\inf_{v_k'\to\bar v}\limsup_{k\to+\infty} t_k^{-1}\left[f\left(x_k'+t_k v_k'\right)-f\left(x_k'\right)\right]\\ &= \alpha + d^\uparrow f(\bar x;\bar v).\end{aligned}$$

Finally, we take the supremum $\sup\limits_{x_k\to_f\bar x}$ in the left-hand side in order to obtain the inequality that we need.

Corollary 2 *If the lsc function* $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ *is convex, then* $d^- f$ *is epi-continuous, i.e.,*

$$d^- f(\bar x;\bar v) = \lim\sup_{x\to_f \bar x}\inf_{v\to\bar v} d^- f(x;v),$$

for all $\bar x\in \operatorname{dom} f$ *and* $\bar v\in X$.

Proof It is direct from Proposition 6 and the equality $d^- f(\bar x;\bar v) = d^\uparrow f(\bar x;\bar v)$ for convex functions.

If in the above results we put as lsc function the indicator function of a closed set, we get the following results.

Corollary 3 ([6,10,18,24]) *Let* $S\subset X$ *be a closed set and* $\bar x\in S$. *Then,*

(i) *if* $\dim X < +\infty$ *one has*

$$\liminf_{x\to_S\bar x} T_B(S;x) = T_C(S;\bar x); \tag{32}$$

(ii) *if* S *is epi-Lipschitzian or convex, one has the same equality* (32);
(iii) *if* X *is reflexive, one has*

$$\liminf_{x\to_S\bar x} T_B^w(S;x) = T_C(S;\bar x). \tag{33}$$

Proof It is similar to that of Corollary 1, using Propositions 2, 3, 5, and 6 and recalling (4), (5), and (6).

Finally, we prove that the equality (25) holds for every lsc compactly epi-Lipschitzian function. We point out that lsc functions on finite dimensional spaces and directionally Lipschitzian functions are compactly epi-Lipschitzian. Separate proofs of (25) have been included for a better understanding and for readers who are not interested in the heavy development of compactly epi-Lipschitzian functions for that equality.

We begin with the definition of a new directional derivative directly related to a tangent cone defined by Borwein and Strójwas in [5].

Definition 3 Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function. For all $\bar{x} \in \operatorname{dom} f$ and $\bar{v} \in X$ we define

$$\begin{aligned} f^{\mathcal{K}}(\bar{x};\bar{v}) &:= \sup_{\delta>0} \inf_{K\in\mathcal{K}(\bar{v})} \inf_{\varepsilon>0} \sup_{\substack{x\in B_f(\bar{x},\varepsilon)\\ t\in]0,\varepsilon[}} \inf_{v\in B[\bar{v},\delta]\cap K} t^{-1}[f(x+tv)-f(x)] \\ &= \sup_{\delta>0} \inf_{\substack{K\subset B(0,\delta)\\ K \text{ compact}}} \inf_{\varepsilon>0} \sup_{\substack{x\in B_f(\bar{x},\varepsilon)\\ t\in]0,\varepsilon[}} \inf_{z\in K} t^{-1}[f(x+t\bar{v}+tz)-f(x)] \\ &= \sup_{\delta>0} \inf_{\substack{K\subset B(0,\delta)\\ K \text{ compact}}} \sup_{\substack{x_k\to_f \bar{x}\\ t_k\to 0^+}} \limsup_{k\to+\infty} \inf_{z\in K} t_k^{-1}[f(x_k+t_k\bar{v}+t_kz)-f(x_k)], \end{aligned}$$

where $\mathcal{K}(\bar{v}) = \{K \subset X \text{ compact } : \bar{v} \in K\}$.

Remark 1 From the above definition, we observe the following facts:

(i) $d^{\uparrow} f \leq f^{\mathcal{K}}$;
(ii) if $\dim X < +\infty$, then $d^{\uparrow} f = f^{\mathcal{K}}$.

Following an analogous proof to that of Proposition 2 we obtain the following result.

Proposition 7 *For any lsc function f and for all $\bar{x} \in \operatorname{dom} f$ and $\bar{v} \in X$ one has*

$$\limsup_{x\to_f \bar{x}} \inf_{v\to\bar{v}} d^- f(x;v) \leq f^{\mathcal{K}}(\bar{x};\bar{v}). \tag{34}$$

Proof We follow the proof of Proposition 2 taking any $K \in \mathcal{K}(\bar{v})$. Noting that $B[\bar{v},\delta] \cap K$ is compact, we change (27) by

$$\exists\, v_1,\ldots,v_n \text{ in } B[\bar{v},\delta] \quad \text{such that } B[\bar{v},\delta]\cap K \subset \bigcup_{i=1}^{n} B(v_i,\xi(x,v_i)).$$

If we take $\hat{v} \in \{v_i : i = 1,\ldots,n\}$ such that $d^- f(x;\hat{v}) = \min\{d^- f(x;v_i) : i = 1,\ldots,n\}$ and $\eta = \min\{\varepsilon, \xi(x,v_1),\ldots,\xi(x,v_n)\}$, we obtain

$$\begin{aligned} \inf_{v\in B[\bar{v},\delta]} d^- f(x;v) &\leq d^- f(x;\hat{v}) \\ &\leq \alpha + \inf_{v'\in B[\bar{v},\delta]\cap K} t^{-1}[f(x+tv')-f(x)] \quad \forall\, t \in]0,\eta] \end{aligned}$$

and hence

$$\sup_{x\in B_f(\bar{x},\varepsilon)} \inf_{v\in B[\bar{v},\delta]} d^-(x;v) \le \alpha + \sup_{\substack{x'\in B_f(\bar{x},\varepsilon)\\ 0<t\le\varepsilon}} \inf_{v'\in B[\bar{v},\delta]\cap K} t^{-1}\left[f\left(x'+tv'\right)-f\left(x'\right)\right],$$

for $\varepsilon,\ \delta > 0, \alpha > 0$, and $K \in \mathcal{K}(\bar{v})$ arbitrary, which proves the desired result.

Our objective now is to prove $d^{\uparrow} f = f^{\mathcal{K}}$ when f is compactly epi-Lipschitzian. Recall that a lsc function f is compactly epi-Lipschitzian (see [16]) at $\bar{x} \in \operatorname{dom} f$ provided there exists a compact set $K \subset X$ such that

$$f^{\square}(\bar{x};K) := \limsup_{\substack{x\to_f \bar{x}\\ t\to 0^+\\ b\to 0}} \inf_{z\in K} t^{-1}[f(x+tb+tz)-f(x)] < +\infty. \tag{35}$$

In order to prove the equality $d^{\uparrow} f = f^{\mathcal{K}}$ for compactly epi-Lipschitzian functions, we need the two lemmas below.

Lemma 2 *Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lsc function, $\bar{x} \in \operatorname{dom} f$, and $\bar{v} \in X$ such that $d^{\uparrow} f(\bar{x};\bar{v}) < +\infty$. Then, for all $\varepsilon > 0$, $x_k \to_f \bar{x}$, and $t_k \to 0^+$ there exists $v_k \to \bar{v}$ such that*

$$\limsup_{k\to+\infty} t_k^{-1}[f(x_k+t_kv_k)-f(x_k)] < d^{\uparrow} f(\bar{x};\bar{v}) + \varepsilon, \tag{36}$$

and $x_k + t_kv_k \to_f \bar{x}$.

Proof The existence of $v_k \to \bar{v}$ satisfying (36) is direct from the definition of $d^{\uparrow} f$. For k large enough one has

$$t_k^{-1}[f(x_k+t_kv_k)-f(x_k)] \le d^{\uparrow} f(\bar{x};\bar{v}) + \varepsilon,$$

hence $f(x_k+t_kv_k) \le t_k(d^{\uparrow} f(\bar{x};\bar{v})+\varepsilon) + f(x_k)$ and we conclude due to the lower semicontinuity of f that $f(x_k+t_kv_k) \to f(\bar{x})$.

The statement of the second lemma is easily verified.

Lemma 3 *Let $f^{\square}(\bar{x};K)$ be the quantity defined in (35). Then*

$$f^{\square}(\bar{x};K) < \beta \Leftrightarrow f^{\square}(\bar{x};\alpha K) < \alpha\beta \qquad \textit{for all } \alpha > 0.$$

Theorem 5 *If the lsc function f is compactly epi-Lipschitzian at $\bar{x} \in \operatorname{dom} f$, then*

$$d^{\uparrow} f(\bar{x};\bar{v}) = f^{\mathcal{K}}(\bar{x};\bar{v}) \quad \textit{for all } \bar{v} \in X.$$

Proof Fix any $\bar{v} \in \operatorname{dom} d^{\uparrow} f(\bar{x};\cdot)$. We only must prove $d^{\uparrow} f(\bar{x};\bar{v}) \ge f^{\mathcal{K}}(\bar{x};\bar{v})$. Let $\beta \in \mathbb{R}$, K a compact set in X, and $R > 0$ such that $f^{\square}(\bar{x};K) < \beta$ and $K \subset B(0,R)$.

Take $\delta,\ \varepsilon > 0$, $x_k \to_f \bar{x}$ and $t_k \to 0^+$. From Lemma 2 there exists $v_k \to \bar{v}$ which satisfies (36) and $x_k + t_k v_k \to_f \bar{x}$. Since

$$\begin{aligned}\inf_{z\in(\delta/2R)K} t_k^{-1}[f(x_k+t_kv+t_kz)-f(x_k)] &= t_k^{-1}[f(x_k+t_kv_k)-f(x_k)]\\ &+\inf_{z\in(\delta/2R)K} t_k^{-1}[f(x_k+t_kv_k+t_kz-t_k(v_k-v))-f(x_k+t_kv_k)],\end{aligned}$$

taking the upper limit $\limsup\limits_{k\to+\infty}$ and using Lemma 3, one obtains

$$\begin{aligned}\limsup_{k\to+\infty}\inf_{z\in(\delta/2R)K} t_k^{-1}[f(x_k+t_kv+t_kz)-f(x_k)] &\le d^{\uparrow}f(\bar{x};\bar{v})+\varepsilon+f^{\square}(\bar{x};(\delta/2R)K)\\ &\le d^{\uparrow}f(\bar{x};\bar{v})+\varepsilon+(\delta/2R)\beta,\end{aligned}$$

for all $x_k \to_f \bar{x}$ and $t_k \to 0^+$. Since $(\delta/2R)K \subset B(0,\delta)$ we have

$$\begin{aligned}&\inf_{\substack{\tilde{K}\subset B(0,\delta)\\ \tilde{K}\text{ compact}}}\ \sup_{\substack{x_k\to_f\bar{x}\\ t_k\to 0^+}}\ \limsup_{k\to+\infty}\ \inf_{z\in\tilde{K}} t_k^{-1}[f(x_k+t_kv+t_kz)-f(x_k)]\\ &\le d^{\uparrow}f(\bar{x};\bar{v})+\varepsilon+(\delta/2R)\beta,\end{aligned}$$

for $\delta,\ \varepsilon > 0$ arbitrary, which proves the desired inequality.

Theorem 6 *If the lsc function f is compactly epi-Lipschitzian at $\bar{x}\in$ dom f, then*

$$\lim\ \sup_{x\to_f\bar{x}}\ \inf_{v\to\bar{v}} d^{-}f(x;v) = d^{\uparrow}f(\bar{x};\bar{v}),$$

for all $\bar{v}\in X$.

Proof It is direct from Theorems 2 and 5, and Proposition 7.

In [5] Borwein and Strojwas introduced the following tangent cone of a closed set $S\subset X$ at $\bar{x}\in S$

$$F(S;\bar{x}) := \bigcap_{\delta>0}\bigcup_{\varepsilon>0}\bigcup_{\substack{K\subset B(0,\delta)\\ K\text{ compact}}}\bigcap_{\substack{x\in B(\bar{x},\varepsilon)\cap S\\ t\in]0,\varepsilon[}}\left[t^{-1}(S-x)+K\right].$$

We can observe almost directly that

$$\Psi_{F(S;\bar{x})}(\bar{v}) = (\Psi_S)^{\mathcal{K}}(\bar{x};\bar{v}) \tag{37}$$

and

$$(\Psi_{\text{epi } f})^{\mathcal{K}}(\bar{x}, f(\bar{x});\bar{v},s) = (\Psi_{\text{epi } f^{\mathcal{K}}(\bar{x};\cdot)})(\bar{v},s), \tag{38}$$

which allows us to obtain the following result given in [5].

Corollary 4 ([5]) *If the closed set $S \subset X$ is compactly epi-Lipschitzian at $\bar{x} \in S$, then*

$$\liminf_{x \to_S \bar{x}} T_B(S; x) = F(S; \bar{x}) = T_C(S; \bar{x}).$$

Proof It is direct from (37) and (38) and Theorem 6. Recall that a lsc function is compactly epi-Lipschitzian iff its epigraph is a compactly epi-Lipschitzian set.

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